J. Fluid Mech. (1972), *vol.* 56, *part* 4, *pp.* 657–681
Printed in Great Britain

Asymptotic analysis of turbulent channel and boundary-layer flow

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(Received 19 January 1972 and in revised form 12 May 1972)

Asymptotic expansion techniques are used, in the limit of large Reynolds number, to study the structure of fully turbulent shear layers. The relevant Reynolds number characterizes the ratio of the local turbulent stress to the local laminar stress, so that a relatively thick outer defect layer, in which, to lowest order, there is a balance between turbulent stress and convection of momentum, may be distinguished from a relatively thin wall layer, in which, to lowest order, there is a balance between turbulent and laminar stresses. The two cases examined are channel (or pipe) flow and two-dimensional boundary-layer flow with an applied pressure gradient, upstream of any separation. Attention, for these two cases, is confined to the flow of incompressible constant property fluids. Closure is effected through the introduction of an eddy-viscosity model formulated with sufficient generality for most existing models to be special cases. Results are carried to higher orders of approximation to indicate what properties for the friction velocity, integral thicknesses, and velocity profiles, and what conditions for similarity are implied by current eddy-viscosity closures.

1. Introduction

Engineering results for turbulent shear layers can be obtained from solutions of time-averaged conservation equations (cf., for example, Townsend 1956). To make these equations a fully determined system, one must introduce closure conditions, which relate the Reynolds turbulent stresses, the time averages of the fluctuating quantities, to mean flow quantities and their derivatives. While some important general lowest order results may be derived without the introduction of these closure conditions, complete results to lowest order (and to higher orders) require closure. This paper concerns the extraction of results for turbulent shear layers for a particular postulated closure condition in the asymptotic limit of large (turbulent) Reynolds number.

For the incompressible constant property fluids considered in this paper, the Reynolds number for *laminar* flow characterizes the ratio of convective to diffusive forces. A large Reynolds number implies that, near the boundary, a relatively thin inner diffusive flow layer describable by a parabolic boundary-value problem may be distinguished from a relatively thick outer potential flow layer describable by an elliptic boundary-value problem (cf., for example, Van Dyke 1964; Cole 1968).

It is important to note that, for turbulent shear flows near a boundary, an inviscid potential outer flow layer is separated from a relatively thin shear layer near the wall, by adopting (classical) boundary-layer approximations, not by formal justification, but, rather, on an intuitive and empirical basis (Rotta 1962). The Reynolds number exploited in the asymptotic analysis of *turbulent* shear flows near a wall characterizes the ratio of the local turbulent stress to the local laminar stress. It has long been known, in a general way, that, when this Reynolds number becomes large, the relatively thin shear flow layer (or boundary layer) near the wall may itself be further subdivided into two sublayers (Coles 1969). One sublayer, the defect layer, is a thicker exterior portion, in which (i) the tangential velocity may be expressed as a small perturbation to the 'outer edge' value as the Reynolds number gets large, and (ii) the laminar stress is small relative to the turbulent stress and the convective and pressure-gradient terms. The other sublayer, the wall layer, is a thinner interior portion, in which the turbulent and laminar stress terms are of comparable magnitude, but the convective and pressure-gradient terms are (often) of higher order. The systematic exploitation of these properties is here carried out by means of limit-process expansion techniques. In the generality of results, and in a myriad of details, the results obtained differ from those of earlier intuitive approaches (Mellor & Gibson 1966; Mellor 1966).

The closure, postulated here for the turbulent boundary-value problem, is presented in terms of the (conventional) phenomenological device, the eddy viscosity. It has been asserted, but not universally (cf. Mellor & Herring 1971), that the eddy viscosity, as a local algebraic relationship, suffices for equilibrium (or self-similar) turbulent flows only, while a differential relationship (emphasizing the history) is required for treatment of non-equilibrium (or non-self-similar) flows (cf. Bradshaw 1967). The point of view adopted here is that the former eddy-viscosity concept remains a viable one that deserves (asymptotic) analysis, after which the same approach should be applied to the latter concept. In fact, the authors believe that the asymptotic apparatus developed in this paper will need little extension to be applicable to most of the (so-called) second-order (or field method) closures now in use.

A more practical problem with the eddy-viscosity approach is that, with so many forms having been proposed for this quantity, a great proliferation of solutions by numerical and integral techniques has appeared. A major goal of the current work is to delineate what may be expected from this approach, through the analytic treatment of a two-layer eddy-viscosity model formulated with enough generality to encompass most of the models proposed. The proposed asymptotic expansion techniques, in the limit of large Reynolds number, require only that a few, conventional, rather weak statements be made concerning the properties of the eddy viscosity at the wall, in the overlap region (intermediate to the wall and defect layers), and at the 'outer edge'. By means of these techniques, it is then deduced what such a general eddy-viscosity model implies about the friction velocity, integral thicknesses, conditions for equilibrium flow, etc., to successive orders of approximation. The method obtains these results by using the knowledge of the properties peculiar to the wall and the defect sublayers in a way that no numerical integration can. Further, this method succeeds best in the singular limits that provide difficulties for the numerical methods.

Previous attempts at the application of such expansion techniques to turbulent shear layers near walls have dealt mainly with the reconstruction of the general lowest order classical results, valid independently of closure (Gill 1968; Tennekes 1968; Yajnik 1970; Mellor 1971). While, in the attainment of even this limited goal, certain unnecessary compromises have been accepted by previous workers, the interest here, as was previously noted, is in obtaining new results by solving the boundary-value problem of the turbulent boundary layer for a broad class of closures (see Saffman (1970) for an introductory discussion of the potentialities of the employment of these expansion techniques for obtaining solutions for explicit closures).

Solutions are developed in § 2 for turbulent channel (or pipe) flow and, in § 3, for two-dimensional turbulent boundary-layer flow with an applied pressure gradient, upstream of any separation. For the case of channel flow an eddy viscosity independent of the local mean strain rate (cf., for example, Mellor 1966) is treated in the main text while an eddy viscosity linearly proportional to the local mean strain rate (cf., for example, Mellor 1966) is considered in the appendix. For the case of boundary-layer flow, an eddy viscosity independent of the local mean strain rate is treated.

2. Turbulent channel flow

2.1. *The equations of motion*

Consider the steady two-dimensional turbulent flow of a fluid of constant density and viscosity (i.e., ρ^*, ν^* = constant) between two parallel, plane, smooth, stationary walls of infinite extent. Let

$$x^* = h^*x, \quad y^* = h^*y \tag{2.1.1}$$

represent the co-ordinates tangential and normal to the plane of symmetry of the channel, respectively, with h^* the half-depth of the channel. The mean velocity components in the x^* and y^* directions and the mean axial pressure gradient are

$$u^* = u^*(y^*) = u_0^*\,u(y), \quad v^* \equiv 0, \tag{2.1.2a}$$

$$dp^*/dx^* = (\rho^*u_0^{*2}/h^*)\,dp/dx = \text{constant} < 0, \tag{2.1.2b}$$

where u_0^* is the mean velocity in the plane of symmetry. The relevant Newton laminar stress, Reynolds turbulent stress and total stress are, respectively,

$$\tau_l^* = \tau_l^*(y^*) = \rho^*\nu^*\frac{du^*}{dy^*} = \frac{\rho^*u_0^{*2}}{u_0^*h^*/\nu^*}\frac{du}{dy}, \tag{2.1.3a}$$

$$\tau_t^* = \tau_t^*(y^*) = \rho^*(-\overline{u^{*\prime}v^{*\prime}}) = \frac{\rho^*u_0^{*2}}{u_0^*h^*/\nu^*}\,\epsilon\,\frac{du}{dy}, \tag{2.1.3b}$$

$$\tau^* = \tau^*(y^*) = \tau_l^*(y^*)+\tau_t^*(y^*) = \rho^*u_0^{*2}\,\tau(y) = \frac{\rho^*u_0^2}{u_0^*h^*/\nu^*}\{1+\epsilon\}\frac{du}{dy}, \tag{2.1.3c}$$

where ϵ is the non-dimensional kinematic eddy viscosity. In the analysis that follows, it is assumed that the channel Reynolds number $Re = u_0^*h^*/\nu^* \to \infty$.

In the domain $-\infty < x < \infty$, $0 \leqslant y \leqslant 1$, the non-dimensional boundary-value problem describing this channel flow is taken to be

$$\frac{dp}{dx} = \frac{1}{Re}\frac{d}{dy}\left(\{1+\epsilon\}\frac{du}{dy}\right) = \text{constant} < 0; \tag{2.1.4}$$

$$u \to 0, \quad \epsilon \to 0 \quad \text{as} \quad y \to 1, \quad u \to 1, \quad \frac{1}{Re}\{1+\epsilon\}\frac{du}{dy} \to 0 \quad \text{as} \quad y \to 0. \tag{2.1.5}$$

Integration of (2.1.4) over the domain of y, subject to the conditions (2.1.5), yields

$$\frac{dp}{dx} = \frac{1}{Re}\left(\frac{du}{dy}\right)_{y=1} = -u_\tau^2, \tag{2.1.6}$$

where $u_\tau = \{-(1/Re)(du/dy)_{y=1}\}^{\frac{1}{2}}$ is the (positive) non-dimensional friction velocity at the wall ($y = 1$). Further, from the above, it is seen that (2.1.4) has a first integral, which may be expressed as

$$\frac{1}{Re}\{1+\epsilon\}\frac{du}{dy} + u_\tau^2 y = 0. \tag{2.1.7}$$

2.2. *The eddy-viscosity model*

In order to solve the foregoing boundary-value problem, the kinematic eddy viscosity ϵ must be specified. The eddy viscosity adopted here (cf. a more complete discussion of the model in § 3.2) is given by

$$\epsilon(y; Re) = \epsilon(\eta, \zeta; \xi) = \xi M(\eta) N(\zeta),$$

where $\quad \xi = \{Re\, u_\tau(Re)\}, \quad \eta = \{1-y\}, \quad \zeta = \xi\eta = \{Re\, u_\tau(Re)\}\{1-y\}. \qquad (2.2.1)$

Note that, in terms of the (normal) co-ordinates η and ζ,

$$u_\tau = u_\tau(Re) = \{(1/Re)(du/d\eta)_{\eta=0}\}^{\frac{1}{2}} = (du/d\zeta)_{\zeta=0}.$$

It is assumed, subject to verification, that

$$u_\tau(Re) \to 0 \quad \text{and} \quad \xi(Re) = \{Re\, u_\tau(Re)\} \to \infty \quad \text{as} \quad Re \to \infty.$$

Thus, for $\xi \to \infty$ with η fixed, $\zeta = \xi\eta \to \infty$; while for $\xi \to \infty$ with ζ fixed, $\eta = \zeta/\xi \to 0$. For the development presented, the functions $M(\eta)$ and $N(\zeta)$, respectively, need have only the following asymptotic forms:

$$\left.\begin{aligned} &M(\eta) = \kappa\eta M_0(\eta) \to \kappa\eta(1+\kappa_2\eta^2+\ldots) \to 0 \quad \text{as} \quad \eta \to 0 \\ &\qquad\qquad\qquad\qquad (\kappa, \kappa_2, \ldots = \text{constants of } O(1)), \\ &M(\eta) \to K_1[1-O(\{1-\eta\}^n)] \to K_1 \text{ as } \eta \to 1 \ \ (K_1 = \text{constant of } O(1)); \end{aligned}\right\} \tag{2.2.2a}$$

$$\left.\begin{aligned} &N(\zeta) = N_0(\zeta) \to C\zeta^2 + \ldots \to 0 \text{ as } \zeta \to 0 \ \ (C, \ldots = \text{constants of } O(1)), \\ &N(\zeta) = N_0(\zeta) \to [1-O(\exp\{-\zeta\})] \to 1 \quad \text{as} \quad \zeta \to \infty. \end{aligned}\right\} \tag{2.2.2b}$$

2.3. *The basic formulation*

For the analysis, based on the given eddy-viscosity model, consider the following restatement of the boundary-value problem. The governing differential equation of (2.1.7), in terms of the variables of (2.2.1) and (2.2.2), becomes

$$\{[\kappa\eta M_0(\eta)\, N_0(\xi\eta)] + (1/\xi)\}\, du(\eta; \xi)/d\eta = u_\tau(\xi)\{1-\eta\}, \tag{2.3.1a}$$

or, alternatively,

$$\{1+\kappa\zeta N_0(\zeta)\, M_0(\zeta/\xi)\}\, du(\zeta,\xi)/d\zeta = u_\tau(\xi)\{1-(\zeta/\xi)\}. \tag{2.3.1b}$$

The (relevant) boundary conditions of (2.1.5) reduce to

$$\begin{aligned} u(\eta;\xi) = u(\zeta;\xi) &\to 0 \quad \text{as} \quad \eta,\zeta \to 0, \\ u(\eta;\xi) = u(\zeta;\xi) &\to 1 \quad \text{as} \quad \eta \to 1,\ \zeta \to \xi \to \infty. \end{aligned} \tag{2.3.2}$$

2.4. *The defect layer*

For the defect layer, the spatial and velocity variables, respectively, are taken to be

$$\eta = 1-y, \tag{2.4.1a}$$

$$f(\eta;\xi) = [1-u(y;Re)]/u_\tau(Re),$$

so that $$u(\eta;\xi) = 1-u_\tau(\xi)f(\eta;\xi); \quad u_\tau(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{2.4.1b}$$

The boundary-value problem in defect-layer variables, for $0 < \eta \leqslant 1$, is given by

$$\left\{M(\eta)\, N(\xi\eta) + \frac{1}{\xi}\right\}\frac{df(\eta;\xi)}{d\eta} + 1 - \eta = \left\{\kappa\eta M_0(\eta)\, N_0(\xi\eta) + \frac{1}{\xi}\right\}\frac{df(\eta;\xi)}{d\eta} + 1 - \eta = 0;$$

$$f(\eta;\xi) \to 0 \quad \text{as} \quad \eta \to 1. \tag{2.4.2}$$

From (2.2.1) and (2.2.2), it is seen that the defect-layer viscosity function $\{M(\eta)\,N(\xi\eta)+(1/\xi)\} \to \{M(\eta)+(1/\xi)\} \to M(\eta)$ as $\xi \to \infty$ with η fixed. This limit for the viscosity function implies that the laminar stress is negligible compared with the turbulent stress in such a defect layer.

Consider that the defect velocity function $f(\eta;\xi)$ is taken to have an asymptotic expansion of the form

$$f(\eta;\xi) \cong \sum_{n=0} S_n(\xi) f_n(\eta) = f_0(\eta) + \sum_{n=1} S_n(\xi) f_n(\eta), \tag{2.4.3}$$

with the gauge functions $S_n(\xi)$ ordered such that $S_{n+1}(\xi)/S_n(\xi) \to 0$ as $\xi \to \infty$.

To leading order of approximation, then, substitution of (2.4.3) into (2.4.2) yields

$$f_0'(\eta) = -\frac{1-\eta}{M(\eta)} = -\frac{1-\eta}{\kappa\eta M_0(\eta)}; \quad f_0(1) = 0. \tag{2.4.4a}$$

The solution of (2.4.4*a*) is determined to be

$$f_0(\eta) = \kappa^{-1}[f_{00}(\eta) - f_{01}(\eta)],$$

where $$f_{00}(\eta) = \frac{\log(1/\eta)+\eta}{M_0(\eta)}, \quad f_{01}(\eta) = \frac{\kappa}{K_1} + \int_\eta^1 \left[\frac{M_0'(t)}{M_0(t)}\right] f_{00}(t)\, dt. \tag{2.4.4b}$$

With $S_1 = 1/\xi$, the equation and boundary condition for $f_1(\eta)$ are

$$f_1'(\eta) = \frac{1-\eta}{\{M(\eta)\}^2} = \frac{1-\eta}{\{\kappa\eta M_0(\eta)\}^2}; \quad f_1(1) = 0. \tag{2.4.5a}$$

The solution of (2.4.5a) is

$$f_1(\eta) = -\kappa^{-2}[f_{10}(\eta) - f_{11}(\eta)],$$

where $$f_{10}(\eta) = \frac{1/\eta - \log(1/\eta)}{\{M_0(\eta)\}^2}, \quad f_{11}(\eta) = \left(\frac{\kappa}{K_1}\right)^2 - 2\int_\eta^1 \left[\frac{M_0'(t)}{M_0(t)}\right] f_{10}(t)\,dt. \quad (2.4.5b)$$

Note that, with $S_n = (1/\xi)^n$, the equation and boundary condition for $f_n(\eta)$ are

$$f_n'(\eta) = \frac{(-1)^{n+1}(1-\eta)}{\{M(\eta)\}^{n+1}} = \frac{(-1)^{n+1}(1-\eta)}{\{\kappa\eta M_0(\eta)\}^{n+1}}; \quad f_n(1) = 0. \quad (2.4.6)$$

Thus, the asymptotic form for the defect-layer velocity is

$$u(\eta;\xi) \cong 1 - u_\tau(\xi)\{f_0(\eta) + (1/\xi) f_1(\eta) + \ldots\}, \quad (2.4.7)$$

where $f_0(\eta)$ and $f_1(\eta)$ are given by (2.4.4b) and (2.5.4b), respectively. As $\eta \to 1$, it is required that $u(\eta;\xi) \to 1$; while it is determined that, as $\eta \to 0$,

$$u(\eta;\xi) \to 1 - \frac{1}{\kappa} u_\tau(\xi)\left\{\left[\log\left(\frac{1}{\eta}\right) - w_0 + \eta + \ldots\right] - \frac{1}{\kappa\xi}\left[\frac{1}{\eta} - \log\left(\frac{1}{\eta}\right) + w_1 + \ldots\right] + \ldots\right\}, \quad (2.4.8)$$

where $w_0 = f_{01}(0)$, $w_1 = -f_{11}(0), \ldots$ are constants.

To examine the behaviour of u, based on this defect-layer expansion, in a region between the defect and wall layers, the intermediate variable λ, defined by

$$\lambda = \eta/\phi(\xi) = \zeta/\{\xi\phi(\xi)\}; \quad \phi(\xi) \to 0, \quad \xi\phi(\xi) \to \infty \quad \text{as} \quad \xi \to \infty, \quad (2.4.9)$$

is introduced. Then it follows that the defect-layer variable η satisfies

$$\eta = \phi(\xi)\lambda \to 0 \quad \text{as} \quad \xi \to \infty, \quad \lambda \text{ fixed}. \quad (2.4.10)$$

Application of the intermediate limit, λ fixed, $\xi \to \infty$, to (2.4.8) yields

$$u \cong 1 - \frac{1}{\kappa} u_\tau \left\{[\log\left\{\frac{1}{\phi\lambda}\right\} - w_0 + \phi\lambda + \ldots] - \frac{1}{\kappa\xi}\left[\frac{1}{\phi\lambda} - \log\left\{\frac{1}{\phi\lambda}\right\} + w_1 + \ldots\right] + \ldots\right\}. \quad (2.4.11)$$

2.5. *The wall layer*

For the wall layer, the spatial and velocity variables, respectively, are taken to be

$$\zeta = \xi\eta = \{Re\, u_\tau(Re)\}\{1-y\}, \quad (2.5.1a)$$

$$\left.\begin{aligned} g(\zeta;\xi) &= u(y;Re)/u_\tau(Re), \\ \text{so that} \quad u(\zeta;\xi) &= u_\tau(\xi)\, g(\zeta;\xi); \quad u_\tau(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \end{aligned}\right\} \quad (2.5.1b)$$

The boundary-value problem in wall-layer variables, for $0 \leqslant \zeta < \infty$, is given by

$$\left.\begin{aligned} \{1 + \kappa\zeta N_0(\zeta) M_0(\zeta/\xi)\}\, dg(\zeta;\xi)/d\zeta - \{1 - (\zeta/\xi)\} = 0; \\ g(\zeta;\xi) \to 0 \quad \text{as} \quad \zeta \to 0. \end{aligned}\right\} \quad (2.5.2)$$

From (2.2.1) and (2.2.2), it is seen that the wall-layer viscosity function

$$1 + \kappa\zeta N_0(\zeta) M_0(\zeta/\xi) \to 1 + \kappa\zeta N_0(\zeta)$$

as $\xi \to \infty$ with ζ fixed. This limit for the viscosity function implies that the laminar stress and the turbulent stress are of the same order of magnitude in such a layer.

Suppose that the wall velocity function $g(\zeta;\xi)$ has an asymptotic expansion of the form

$$g(\zeta;\xi) \cong \sum_{m=0} T_m(\xi)\, g_m(\zeta) = g_0(\zeta) + \sum_{m=1} T_m(\xi)\, g_m(\zeta), \tag{2.5.3}$$

where the gauge functions $T_m(\xi)$ satisfy $T_{m+1}(\xi)/T_m(\xi) \to 0$ as $\xi \to \infty$. Then, to leading approximation, the flow in the wall layer is prescribed by

$$g_0'(\zeta) = 1/[1+\kappa\zeta N_0(\zeta)]; \quad g_0(0) = 0. \tag{2.5.4a}$$

The solution of (2.5.4a) is found to be

$$g_0(\zeta) = \frac{1}{\kappa}[g_{00}(\zeta)+g_{01}(\zeta)],$$

where $$g_{00}(\zeta) = \log\{1+(\kappa\zeta)\}, \quad g_{01}(\zeta) = \kappa^2 \int_0^\zeta \frac{s\{1-N_0(s)\}\,ds}{\{1+\kappa s\}\{1+\kappa s N_0(s)\}}. \tag{2.5.4b}$$

With $T_1 = 1/\xi$, the equation and boundary condition for $g_1(\zeta)$ are

$$g_1'(\zeta) = -\zeta/[1+\kappa\zeta N_0(\zeta)]; \quad g_1(0) = 0. \tag{2.5.5a}$$

The solution of (2.5.5a) is

$$g_1(\zeta) = -\left(\frac{1}{\kappa}\right)^2 [g_{10}(\zeta)+g_{11}(\zeta)],$$

where

$$g_{10}(\zeta) = (\kappa\zeta) - \log\{1+(\kappa\zeta)\}, \quad g_{11}(\zeta) = -\kappa^3 \int_0^\zeta \frac{s^2\{1-N_0(s)\}\,ds}{\{1+\kappa s\}\{1+\kappa s N_0(s)\}}. \tag{2.5.5b}$$

Thus, the asymptotic form for the wall-layer velocity is

$$u(\zeta;\xi) \cong u_\tau(\xi)\{g_0(\zeta) + (1/\xi)\, g_1(\zeta) + \dots\}, \tag{2.5.6}$$

where $g_0(\zeta)$ and $g_1(\zeta)$ are given by (2.5.4b) and (2.5.5b), respectively. As $\zeta \to 0$, it is required that $u(\zeta;\xi) \to 0$; while it is found that, as $\zeta \to \infty$,

$$u(\zeta;\xi) \to \frac{1}{\kappa} u_\tau(\xi) \left\{ \left[\log\zeta + j_0 + \frac{1}{\kappa\zeta} + \dots \right] - \frac{1}{\kappa\xi}[\kappa\zeta - \log\zeta - j_1 + \dots] + \dots \right\}, \tag{2.5.7}$$

where $j_0 = -\{\log(1/\kappa) - g_{01}(\infty)\}$, $j_1 = -\{\log(1/\kappa) + g_{11}(\infty)\}, \dots$ are constants.

Again, to examine the behaviour of u, based on this wall-layer expansion, in a region between the defect and wall layers, the intermediate variable λ, defined by (2.4.9), is re-introduced. It is seen that the wall-layer variable ζ obeys

$$\zeta = \{\xi\phi(\xi)\}\lambda \to \infty \quad \text{as} \quad \xi \to \infty, \quad \lambda \text{ fixed}. \tag{2.5.8}$$

Application of the intermediate limit, λ fixed, $\xi \to \infty$, to (2.5.7) yields

$$u \cong \frac{1}{\kappa} u_\tau \left\{ \left[\log\xi - \log\left\{\frac{1}{\phi\lambda}\right\} + j_0 + \frac{1}{\kappa\xi}\left\{\frac{1}{\phi\lambda}\right\} + \dots \right] \right.$$
$$\left. - \frac{1}{\kappa\xi}\left[(\kappa\xi)(\phi\lambda) - \log\xi + \log\left\{\frac{1}{\phi\lambda}\right\} - j_1 + \dots \right] + \dots \right\}. \tag{2.5.9}$$

Note that (2.5.9) may also be written as

$$u \cong \left[1-\frac{1}{\kappa}u_\tau\left\{\left[\log\left\{\frac{1}{\phi\lambda}\right\}-w_0+\phi\lambda+\ldots\right]-\frac{1}{\kappa\xi}\left[\left\{\frac{1}{\phi\lambda}\right\}-\log\left\{\frac{1}{\phi\lambda}\right\}+w_1+\ldots\right]+\ldots\right\}\right]$$
$$-\left[1-\frac{1}{\kappa}u_\tau\left\{[\log\xi+c_0]+\frac{1}{\kappa\xi}[\log\xi+c_1]+\ldots\right\}\right], \quad (2.5.10)$$

where $c_0 = (j_0-w_0)$, $c_1 = (j_1-w_1)$, ... are constants.

2.6. *Results*

From a comparison of (2.4.11) and (2.5.10), it is seen that the postulated defect- and wall-layer solutions for the velocity u match when

$$1 \cong \frac{1}{\kappa}u_\tau(\xi)\log\xi\left\{\left[1+\frac{c_0}{\log\xi}\right]+\frac{1}{\kappa\xi}\left[1+\frac{c_1}{\log\xi}\right]+\ldots\right\}. \quad (2.6.1)$$

Since $\xi(Re) = Re\,u_\tau(Re)$, directly from (2.6.1), it is determined that the (preliminary) skin-friction law is

$$\frac{1}{u_\tau(Re)} \cong \frac{1}{\kappa}\left\{[\log\{Re\,u_\tau(Re)\}+c_0]+\frac{1}{\kappa}\frac{1}{Re\,u_\tau(Re)}[\log\{Re\,u_\tau(Re)\}+c_1]+\ldots\right\}. \quad (2.6.2)$$

The (asymptotic) solutions for ξ and u_τ, from the transcendental equation (2.6.1), are

$$\frac{1}{\xi(Re)} \cong \frac{\log(\kappa Re)}{\kappa Re}\left[1-\frac{\log\{\log(\kappa Re)\}-c_0}{\log(\kappa Re)}+\ldots\right] \to 0 \quad \text{as} \quad Re, \log Re \to \infty, \quad (2.6.3a)$$

$$u_\tau(Re) \cong \frac{\kappa}{\log(\kappa Re)}\left[1+\frac{\log\{\log(\kappa Re)\}-c_0}{\log(\kappa Re)}+\ldots\right] \to 0 \quad \text{as} \quad Re, \log Re \to \infty. \quad (2.6.3b)$$

Thus, (2.6.3) demonstrates that the two-layer asymptotic analysis of turbulent channel flow presented is valid in the limit $Re, \log Re \to \infty$.

The (conventional) skin-friction law for turbulent channel flow is presented in terms of the cross-sectional average of the mean velocity U^* rather than the mean velocity in the plane of symmetry, u_0^*. This average velocity is defined by

$$U^* = (1/h^*)\int_0^{h^*} u^*\,dy^* = u_0^*\,U = u_0^*\int_0^1 u\,dy.$$

In order to exhibit the results of the present analysis for the above-mentioned skin-friction law, then, it is necessary to derive an (asymptotic) expression for U. From its definition, it is seen that

$$U \cong \int_{\phi\lambda}^1\left[1-u_\tau(\xi)\left\{f_0(\eta)+\frac{1}{\xi}f_1(\eta)+\ldots\right\}\right]d\eta$$
$$+\int_0^{\xi(\phi\lambda)}\left[\frac{1}{Re(\xi)}\left\{g_0(\zeta)+\left(\frac{1}{\xi}\right)g_1(\zeta)+\ldots\right\}\right]d\zeta, \quad (2.6.4)$$

with λ fixed, $\phi(\xi) \to 0$ and $\xi\phi(\xi) \to \infty$ as $\xi \to \infty$ (cf. equation (2.4.9)). Evaluation of the integrals in (2.6.4), subject to this intermediate limit, yields

$$U \simeq 1 - \frac{1}{\kappa} u_\tau \left\{ v_0 - \frac{1}{\kappa\xi} [\log \xi - v_1] + \dots \right\}; \quad v_0 = \int_0^1 \frac{1-t}{M_0(t)} dt, \dots = \text{constant}.$$

With the introduction of $\overline{u}_\tau = u_\tau/U = u_\tau^*/U^*$ and $\overline{Re} = Re\, U = U^*h^*/\nu^*$, from (2.6.1) and (2.6.5), the skin-friction law can be written as

$$\frac{1}{\overline{u}_\tau(\overline{Re})} \simeq \frac{1}{\kappa} \left\{ [\log \{\overline{Re}\, \overline{u}_\tau(\overline{Re})\} + \overline{c}_0] + \frac{1}{\kappa} \frac{1}{\{\overline{Re}\, \overline{u}_\tau(\overline{Re})\}} [2 \log \{\overline{Re}\, \overline{u}_\tau(\overline{Re})\} + \overline{c}_1] + \dots \right\}, \tag{2.6.5}$$

where

$$\overline{c}_0 = (c_0 - v_0) = (j_0 - \{w_0 + v_0\}), \quad \overline{c}_1 = (c_1 - v_1) = (j_1 - \{w_1 + v_1\}), \dots. \quad (2.6.6)\dagger$$

The skin-friction coefficient is defined as $\overline{c_f}(\overline{Re}) = 2\{\overline{u}_\tau(\overline{Re})\}^2$. Thus,

$$\overline{c_f}(\overline{Re}) \simeq \frac{2\kappa^2}{\{\log(\kappa\overline{Re})\}^2} \left[1 + 2 \frac{\log\{\log(\kappa\overline{Re})\} - \overline{c}_0}{\log(\kappa\overline{Re})} + \dots \right] \quad \text{as} \quad \log \overline{Re} \to \infty. \quad (2.6.7)\ddagger$$

The feature of the analysis presented that most warrants comment is the development of the higher order terms in the defect-layer expansion (2.4.7) and the wall-layer expansion (2.5.6). From this development, it is determined that, in the wall layer, the total stress is constant to lowest order (cf. equation (2.5.4)), and that the first modification (of $O(1/\xi)$) to this total stress changes linearly with the distance from the wall (cf. equation (2.5.5)). Further, it is determined that, in the defect layer, the total stress is given by the turbulent stress to lowest order (cf. equation (2.4.4)), and that the leading contribution of the laminar stress to the total stress is of $O(1/\xi)$ (cf. equation (2.4.5)). Thus, as $\xi \to \infty$, with the higher order terms of $O(1/\xi) \to O\{\log(\kappa Re)/(\kappa Re)\}$, at most, relative to the lowest order terms in both the defect-layer and wall-layer expansions, such terms represent small corrections.

Other forms of the eddy viscosity, based on mixing-length theory (due to Prandtl), in which this eddy viscosity ϵ is taken to be proportional to the local mean rate of strain (du/dy), have been proposed (cf., for example, Schlichting 1968; Mellor 1966). Further, it has been indicated that there exists a wide range of conditions for which such a model and the one employed in this section should yield comparable results. In the appendix, an alternative formulation for an eddy-viscosity model based on mixing-length concepts is treated by asymptotic methods parallel to those of this section. For both models, the results, which are quite similar, indicate that such asymptotic methods (for $Re \to \infty$) provide an effective means of analysing closures, in general.

† The values of the constants $\overline{c}_0, \overline{c}_1, \dots$ can be determined only with the specification of the functions $M(\eta)$, $N(\zeta)$ (and/or $M_0(\eta)$, $N_0(\zeta)$), since w_0 and v_0, w_1 and v_1, ... are defined by definite integrals dependent on the function $M_0(\eta)$, and $j_0, j_1, \dots$ are defined by definite integrals dependent on the function $N_0(\zeta)$.

‡ To leading order(s) of approximation, this expression for $\overline{c_f}$ (as $\log \overline{Re} \to \infty$) depends upon the eddy-viscosity model adopted only through the constant κ. It is only in the higher orders of approximation that $\overline{c_f}$ depends upon the constants $(\overline{c}_0, \overline{c}_1, \dots)$ derived from particular functional forms (to be specified) for $M_0(\eta)$ and $N_0(\zeta)$.

3. Two-dimensional turbulent boundary layers

3.1. *The equations of motion*

Consider now the steady two-dimensional turbulent boundary layer of a fluid of constant density and viscosity (ρ^*, ν^* = constant) at a smooth surface. Let

$$x^* = L^*x, \quad y^* = L^*y \tag{3.1.1}$$

represent the co-ordinates tangential and normal to the surface, respectively, with L^* a characteristic longitudinal body length. The mean velocity components in the x^* and y^* directions and the mean kinematic pressure are

$$u^*(x^*,y^*) = u^*_\infty u(x,y), \quad v^*(x^*,y^*) = u^*_\infty v(x,y), \tag{3.1.2a}$$

$$P^*(x^*) = p^*_\infty + (\rho^* u^{*2}_\infty)\, P(x), \tag{3.1.2b}$$

with u^*_∞ and p^*_∞ the (free-stream) velocity and pressure in the undisturbed region far from the surface; the specified tangential velocity at the 'outer edge' of the boundary layer ($y^* \to \infty$) is taken to be

$$U^*(x^*) = u^*_\infty\, U(x), \tag{3.1.3}$$

with $U\,dU/dx = -dP/dx$. The relevant Newton laminar stress, Reynolds turbulent stress and total stress are

$$\tau^*_l(x^*,y^*) = \rho^*\nu^* \frac{\partial u^*}{\partial y^*} = \frac{\rho^* u^{*2}_\infty}{u^*_\infty L^*/\nu^*}\frac{\partial u}{\partial y}, \tag{3.1.4a}$$

$$\tau^*_t(x^*,y^*) = \rho^*(-\overline{u^{*\prime}v^{*\prime}}) = \frac{\rho^* u^{*2}_\infty}{u^*_\infty L^*/\nu^*}\,\epsilon\frac{\partial u}{\partial y}, \tag{3.1.4b}$$

$$\begin{aligned}\tau^*(x^*,y^*) &= \{\tau^*_l(x^*,y^*) + \tau^*_t(x^*,y^*)\} = (\rho^* u^{*2}_\infty)\,\tau(x,y)\\ &= \frac{\rho^* u^{*2}_\infty}{u^*_\infty L^*/\nu^*}\{1+\epsilon\}\frac{\partial u}{\partial y}.\end{aligned} \tag{3.1.4c}$$

In the analysis that follows, it is assumed that the free-stream Reynolds number $Re = u^*_\infty L/\nu^* \to \infty$.

In the domain $x \geqslant x_0$, $0 \leqslant y < \infty$, the non-dimensional boundary-value problem describing this boundary-layer flow is taken to be

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{so that} \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{3.1.5a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{\partial x} = \frac{1}{Re}\frac{\partial}{\partial y}\left(\{1+\epsilon\}\frac{\partial u}{\partial y}\right); \tag{3.1.5b}$$

$$u, v \to 0, \quad \epsilon \to 0 \quad \text{as} \quad y \to 0 \quad (x > x_0), \tag{3.1.6a}$$

$$u \to U, \quad (1/Re)\{1+\epsilon\}\,\partial u/\partial y \to 0 \quad \text{as} \quad y \to \infty \quad (x \geqslant x_0), \tag{3.1.6b}$$

$$u \to u_0 \quad \text{as} \quad x \to x_0 \quad (y > 0). \tag{3.1.6c}$$

Integration of (3.1.5) over the domain of y, subject to the conditions of (3.1.6), yields the (so-called) von Kármán integral, which may be expressed as

$$\gamma \equiv \frac{u_\tau}{U} = \left\{\frac{d\theta}{dx} + \left(2 + \frac{\delta}{\theta}\right)\left(\frac{\theta}{U}\frac{dU}{dx}\right)\right\}^{\frac{1}{2}}, \tag{3.1.7a}$$

where

$$u_\tau = \left\{\frac{1}{Re}\left(\frac{\partial u}{\partial y}\right)_{y=0}\right\}^{\frac{1}{2}}, \quad \delta = \int_0^\infty \left[1 - \frac{u}{U}\right] dy, \quad \theta = \int_1^\infty \frac{u}{U}\left[1 - \frac{u}{U}\right] dy. \tag{3.1.7b}$$

In (3.1.7), $u_\tau(x)$ $(= u_\tau^*/u_\infty^* = \{[\nu^*(\partial u^*/\partial y^*)_{y=0}]^{\frac{1}{2}}/u_\infty^*\})$ is the non-negative friction velocity at the wall; $\delta(x)$ $(= \delta^*/L^*)$ and $\theta(x)$ $(= \theta^*/L^*)$ are the (conventional) positive displacement and momentum thicknesses.

3.2. *The eddy-viscosity model*

In order to solve the boundary-value problem (3.1.5) with (3.1.6), the eddy viscosity ϵ must be specified. The eddy viscosity adopted here is given by

$$\left.\begin{aligned} &\epsilon(x,y;Re) = \epsilon(\xi,\eta,\zeta) = \xi M(\eta)\,N(\zeta),\\ &\xi = Re^*(x) = \{Re\,U(x)\,\Delta(x)\}\,\Delta'(x),\\ &\eta = y/\Delta(x), \quad \zeta = \xi\eta = Re\,U(x)\,\Delta'(x)\,y. \end{aligned}\right\} \tag{3.2.1†}$$

In (3.2.1), $\Delta(x)$ $(= \Delta^*/L^*)$ is the reference boundary-layer thickness (to be determined), while $Re^*(x)\,(=\{U^*\Delta^*/\nu^*\}\,d\Delta^*/dx^*)$ is the reference boundary-layer Reynolds number, to be determined. Note that, for $\xi\to\infty$ with η fixed, $\zeta = \xi\eta\to\infty$; while, for $\xi\to\infty$ with ζ fixed, $\eta = \zeta/\xi\to 0$. For the development presented, the functions $M(\eta)$ and $N(\zeta)$, respectively, need have only the following asymptotic forms:

$$\left.\begin{aligned} &M(\eta) = \kappa\eta M_0(\eta) \to \kappa\eta(1+\kappa_2\eta^2+\ldots)\to 0 \quad \text{as} \quad \eta\to 0\\ &\qquad\qquad\qquad\qquad (\kappa,\kappa_2,\ldots, = \text{constants of } O(1)),\\ &\mathrm{M}(\eta)\to K_\infty[1-O(\exp\{-\eta\})]\to K_\infty \quad \text{as} \quad \eta\to\infty\\ &\qquad\qquad\qquad\qquad (K_\infty = \text{constant of } O(1)); \end{aligned}\right\} \tag{3.2.2a}$$

$$\left.\begin{aligned} &N(\zeta) = N_0(\zeta)\to C\zeta^m+\ldots\to 0 \quad \text{as} \quad \zeta\to 0\\ &\qquad\qquad\qquad\qquad (C, m, \ldots = \text{constants of } O(1)),\\ &N(\zeta) = N_0(\zeta)\to[1-O(\exp\{-\zeta\})]\to 1 \quad \text{as} \quad \zeta\to\infty. \end{aligned}\right\} \tag{3.2.2b}$$

The model, as defined by (3.2.1) and (3.2.2), is a continuous two-layer model for the eddy viscosity (cf. §2). The quantity $\xi M(\eta)$, as given, represents a generalized outer-layer (or defect-layer) factor for the eddy-viscosity function. The basic form of this defect-layer factor is a modification of that suggested by Clauser (1956). Similarly, the quantity $N(\zeta)$ represents a generalized complementary inner-layer (or wall-layer) factor, whose given form incorporates the essential features of the Van Driest (1956) dissipation factor.

It is appropriate to consider the effective viscosity function

$$1+\epsilon(\xi,\eta,\zeta) = 1+\xi M(\eta)\,N(\zeta)$$

in more detail. For an outer or defect layer, defined by $\xi\to\infty$ with η fixed, such that $\zeta = \xi\eta\to\infty$, this viscosity function is given by

$$\begin{aligned} 1+\xi M(\eta)\,N(\xi\eta) &\simeq 1+\xi M(\eta)[1-O(\exp\{-\xi\eta\})]\\ &\simeq \xi\{K_\infty[1-O(\exp\{-\eta\})]+(1/\xi)\} \quad \text{as} \quad \eta\to\infty \qquad (3.2.3a)\\ &\simeq \xi\{\kappa\eta[1+\kappa_2\eta^2+\ldots]+(1/\xi)\} \quad \text{as} \quad \eta\to 0. \qquad (3.2.3b) \end{aligned}$$

† Superscript asterisks are reserved throughout this paper to denote dimensional quantities; quantities without the superscript asterisk are dimensionless. The four exceptions are the use of the superscript asterisk to distinguish the reference boundary-layer Reynolds-number function introduced in (3.2.1) and three 'conventional' Reynolds-number functions introduced below in (3.6.33), (3.6.39b), and (3.6.40a).

For an inner or wall layer, defined by $\xi \to \infty$ with ζ fixed, with $\eta = \zeta/\xi \to 0$,

$$1+\xi M(\zeta/\xi)\,N(\zeta) \simeq 1+\kappa\zeta N(\zeta)[1+\kappa_2(\zeta/\xi)^2+\ldots]$$

$$\simeq 1+(\kappa C)\,\zeta^{m+1}[1+\ldots] \quad \text{as} \quad \zeta \to 0 \tag{3.2.4a}$$

$$\simeq \kappa\zeta[1+\kappa_2(\zeta/\xi)^2+\ldots]+1 \quad \text{as} \quad \zeta \to \infty. \tag{3.2.4b}$$

That the model is continuous follows from the asymptotic behaviour of (i) the outer form as $\eta \to 0$ and (ii) the inner form as $\zeta \to \infty$. Note that, from the above, (i) $\epsilon \to K_\infty \xi$ at the 'outer edge' of the boundary layer (cf. Clauser 1956) and (ii) $\epsilon \to (\kappa C)\,\zeta^{m+1} \to 0$ at the wall (cf. the arguments of Phillips (1969) for $m+1 = 3$).

It should be pointed out that, since $\tau_t^*/\tau_l^* = \tau_t/\tau_l = \epsilon$, because of the (specified) behaviour of this continuous two-layer effective viscosity, (i) in the outer layer, the order of magnitude of the turbulent stress is greater than that of the laminar stress by a factor of ξ ($\epsilon \simeq \xi M(\eta)$); while (ii) in the inner layer, the orders of magnitude of the turbulent and laminar stresses are the same ($\epsilon \simeq \kappa\zeta N(\zeta)$).

3.3. *The basic formulation*

For the analysis, based on the given eddy-viscosity model, consider the following reformulation of the boundary-value problem. The spatial variables are taken to be

$$\xi = Re^*(x), \quad \eta = y/\Delta(x); \quad \xi \to \infty, \quad 0 \leqslant \eta < \infty, \tag{3.3.1a†}$$

so that
$$x\frac{\partial}{\partial x} = \frac{1}{A}\left\{[1-\Pi+B]\,\xi\frac{\partial}{\partial \xi}-\eta\frac{\partial}{\partial \eta}\right\}, \quad y\frac{\partial}{\partial y} = \eta\frac{\partial}{\partial \eta}, \tag{3.3.1b}$$

with
$$A(\xi(x)) = \Delta(x)/x\Delta'(x), \quad B(\xi(x)) = \Delta(x)\,\Delta''(x)/[\Delta'(x)]^2,$$
$$\Pi(\xi(x)) = -\{\Delta(x)/x\Delta'(x)\}\,xU'(x)/U(x).$$

The 'normalized' stream function is taken to be

$$\Psi(\xi,\eta) = \psi(x,y)/U(x)\,\Delta(x), \tag{3.3.2a}$$

so that

$$\left.\begin{aligned} \bar{u}(\xi,\eta) &= u(x,y)/U(x) = \partial\Psi/\partial\eta, \\ \bar{v}(\xi,\eta) &= \frac{v(x,y)}{U(x)\,\Delta'(x)}-\frac{yu(x,y)}{U(x)\,\Delta(x)} = -\left\{[1-\Pi+B]\,\xi\frac{\partial\Psi}{\partial\xi}+[1-\Pi]\Psi\right\}. \end{aligned}\right\} \tag{3.3.2b}$$

The governing differential equations of (3.1.5) in terms of the variables of (3.3.1) and (3.3.2), with the introduction of $E(\xi,\eta) = M(\eta)\,N(\xi\eta)$, become

$$\frac{\partial}{\partial\eta}\left(\left\{E+\frac{1}{\xi}\right\}\frac{\partial^2\Psi}{\partial\eta^2}\right)+[1-\Pi]\Psi\frac{\partial^2\Psi}{\partial\eta^2}-\Pi\left(1-\left[\frac{\partial\Psi}{\partial\eta}\right]^2\right)$$
$$= [1-\Pi+B]\,\xi\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial^2\Psi}{\partial\xi\,\partial\eta}-\frac{\partial\Psi}{\partial\xi}\frac{\partial^2\Psi}{\partial\eta^2}\right), \tag{3.3.3a}$$

† The boundary-layer Reynolds number, the largeness of which is the basis of the succeeding asymptotic expansions, is also a convenient choice for the 'streamwise co-ordinate'. The expansions to be employed thus have the aspects of 'co-ordinate expansions' with the region of validity far downstream from the initial station.

or, alternatively,

$$\frac{\partial}{\partial\eta}\left(\left\{E+\frac{1}{\xi}\right\}\frac{\partial^2\Psi}{\partial\eta^2}-\left[1-\frac{\partial\Psi}{\partial\eta}\right]\left\{[1-\Pi+B]\,\xi\frac{\partial\Psi}{\partial\xi}+[1-\Pi]\Psi\right\}\right)$$
$$=-\left\{[1-\Pi+B]\,\xi\frac{\partial}{\partial\xi}\left(\frac{\partial\Psi}{\partial\eta}\left[1-\frac{\partial\Psi}{\partial\eta}\right]\right)+[1-2\Pi]\left(\frac{\partial\Psi}{\partial\eta}\left[1-\frac{\partial\Psi}{\partial\eta}\right]\right)-\Pi\left[1-\frac{\partial\Psi}{\partial\eta}\right]\right\}. \tag{3.3.3b}$$

The boundary conditions of (3.1.12) become

$$\Psi,\,\partial\Psi/\partial\eta\to 0 \quad\text{as}\quad \eta\to 0,\quad \partial\Psi/\partial\eta\to 1 \quad\text{as}\quad \eta\to\infty. \tag{3.3.4}$$

The von Kármán integral, with the introduction of $S(\xi(x)) = \Delta'(x)$, reduces to

$$\Gamma(\xi(x)) = \frac{\gamma(x)}{\Delta'(x)} = \left\{\frac{1}{\xi S}\left(\frac{\partial^2\Psi}{\partial\eta^2}\right)_{\eta=0}\right\}^{\frac{1}{2}}$$
$$=\left\{[1-\Pi+B]\,\xi\frac{d\Theta}{d\xi}+[(1-3\Pi+B)-(H-1)\,\Pi]\,\Theta\right\}^{\frac{1}{2}}, \tag{3.3.5a}$$

with

$$\left.\begin{aligned}\Lambda(\xi(x)) &= \frac{\delta(x)/\Delta(x)}{\Delta'(x)} = \frac{1}{S}\left(\int_0^\infty\left[1-\frac{\partial\Psi}{\partial\eta}\right]d\eta\right),\\ \Theta(\xi(x)) &= \frac{\theta(x)/\Delta(x)}{\Delta'(x)} = \frac{1}{S}\left(\int_0^\infty\frac{\partial\Psi}{\partial\eta}\left[1-\frac{\partial\Psi}{\partial\eta}\right]d\eta\right),\\ H(\xi(x)) &= \Lambda/\Theta = \delta/\theta.\end{aligned}\right\} \tag{3.3.5b}$$

Note that, from the defining equation for Λ, it follows that there is an additional constraint on the behaviour of Ψ, namely, that

$$\Psi\to\eta-S\Lambda \quad\text{as}\quad \eta\to\infty. \tag{3.3.6}$$

3.4. *The defect layer*

For the defect layer, the spatial variables are taken to be

$$\xi = Re^*(x),\quad \eta = y/\Delta(x);\quad \xi\to\infty,\quad 0<\eta<\infty. \tag{3.4.1}$$

As previously noted, this defect layer is defined by $\xi\to\infty$ with η fixed, so that $\zeta=\xi\eta\to\infty$. The stream function is taken to be

$$\Psi(\xi,\eta) = \psi(x,y)/[U(x)\,\Delta(x)] = \eta - S(\xi)\,F(\xi,\eta), \tag{3.4.2}$$

where $S(\xi)$ is the reference defect function (to be determined). For the analysis, $S(\xi)\to 0$ as $\xi\to\infty$. The velocity components, derivable from this stream function, are

$$\bar{u}(\xi,\eta) = u(x,y)/U(x) = 1-S\,\partial F/\partial\eta, \tag{3.4.3a}$$

$$\bar{v}(\xi,\eta) = \frac{v(x,y)}{U(x)\,\Delta'(x)}-\frac{yu(x,y)}{U(x)\,\Delta(x)} = -[1-\Pi]\left(\eta-\frac{S}{1+T}\left\{\xi\frac{\partial F}{\delta\xi}+F\right\}\right), \tag{3.4.3b}$$

with $T(\xi) = -\xi S'(\xi)/S(\xi)$. It is noted, from (3.4.3a), that

$$(1/\Gamma)\,\partial F/\partial\eta = (1/\gamma)\,[1-(u/U)] = (U-u)/u_\tau,$$

the modified defect velocity law.

The boundary-value problem in the defect-layer variables, from (3.3.3) and (3.3.4), is given by

$$\frac{\partial}{\partial\eta}\left(\left\{\kappa\eta M_0 N_0+\frac{1}{\xi}\right\}\frac{\partial^2 F}{\partial\eta^2}+[1-\Pi]\,\eta\frac{\partial F}{\partial\eta}\right.$$
$$\left.-\frac{1}{1+T}\left\{[1-\Pi]\,\xi\frac{\partial F}{\partial\xi}+[(1-3\Pi)+T(-2\Pi)]\,F\right\}\right)$$
$$=\frac{S}{1+T}\left\{[1-\Pi]\frac{\partial}{\partial\eta}\left(\left\{\xi\frac{\partial F}{\partial\xi}+F\right\}\frac{\partial F}{\partial\eta}\right)\right.$$
$$\left.-[1-\Pi]\,\xi\frac{\partial}{\partial\xi}\left(\left[\frac{\partial F}{\partial\eta}\right]^2\right)-[(1-2\Pi)+T(-1)]\left[\frac{\partial F}{\partial\eta}\right]^2\right\}; \quad (3.4.4)$$

$$F\to\Lambda,\quad \frac{\partial F}{\partial\eta},\frac{\partial^2 F}{\partial\eta^2}\to 0\quad\text{as}\quad \eta\to\infty,\qquad F\to 0\quad\text{as}\quad\eta\to 0. \quad (3.4.5)$$

From (3.4.4) and (3.4.5), it is determined that the first integral of the defect-layer equation is

$$\left\{\kappa\eta M_0 N_0+\frac{1}{\xi}\right\}\frac{\partial^2 F}{\partial\eta^2}+[1-\Pi]\,\eta\frac{\partial F}{\partial\eta}-\frac{1}{1+T}\left\{[1-\Pi]\,\xi\frac{\partial F}{\partial\xi}+[(1-3\Pi)+T(-2\Pi)]\,F\right\}$$
$$-\frac{S}{1+T}\left\{[1-\Pi]\left(\left\{\xi\frac{\partial F}{\partial\xi}+F\right\}\frac{\partial F}{\partial\eta}\right)\right\}=-\frac{1}{1+T}\left\{[1-\Pi]\,\xi\frac{d\Lambda}{d\xi}+[(1-3\Pi)+T(-2\Pi)]\,\Lambda\right\}$$
$$+\frac{S}{1+T}\left\{(1-\Pi)\,\xi\frac{\partial}{\partial\xi}\left(\int_\eta^\infty\left[\frac{\partial F}{\partial\hat\eta}\right]^2 d\hat\eta\right)+[(1-2\Pi)+T(-1)]\left(\int_\eta^\infty\left[\frac{\partial F}{\partial\hat\eta}\right]^2 d\hat\eta\right)\right\}. \quad (3.4.6)$$

3.5. *The wall layer*

For the wall layer, the spatial variables are taken to be

$$\xi = Re\,(x),\quad \zeta=\xi\eta = Re\,(x)\,\{y/\Delta(x)\};\quad \xi\to\infty,\quad 0\leqslant\zeta<\infty. \quad (3.5.1)$$

The wall layer is defined by $\xi\to\infty$, with ζ fixed, so that $\eta=\zeta/\xi\to 0$. The stream function is taken to be

$$\Upsilon(\xi,\zeta)=\psi(x,y)/[U(x)\,\Delta(x)]=(S(\xi)/\xi)\,G(\xi,\zeta), \quad (3.5.2)$$

with $S(\xi)$, now, the reference wall function (to be determined). The velocity components are

$$\bar u(\xi,\zeta)=u(x,y)/U(x)=S\,\partial G/\partial\zeta, \quad (3.5.3a)$$

$$\bar v(\xi,\zeta)=\frac{v(x,y)}{U(x)\,\Delta'(x)}-\frac{yu(x,y)}{U(x)\,\Delta(x)}=\frac{S^2}{\xi S}\frac{[1-\Pi]}{1+T}\left\{\xi\frac{\partial G}{\partial\xi}+\zeta\frac{\partial G}{\partial\zeta}\right\}, \quad (3.5.3b)$$

with, again, $T(\xi)=-\xi S'(\xi)/S(\xi)$. Note that, from (3.5.3$a$), it follows that

$$(1/\Gamma)\,\partial G/\partial\zeta=(1/\gamma)\,(u/U)=u/u_\tau,$$

the modified wall velocity law.

The boundary-value problem, in the wall-layer variables, is given by

$$\frac{\partial}{\partial\zeta}\left(\{1+\kappa\zeta N_0 M_0\}\frac{\partial^2 G}{\partial\zeta^2}\right)-\frac{\Pi}{\xi S}$$
$$=\frac{S^2}{\xi S}\frac{1}{1+T}\left\{-[\Pi+T]\left[\frac{\partial G}{\partial\zeta}\right]^2+[1-\Pi]\,\xi\left(\frac{\partial G}{\partial\zeta}\frac{\partial^2 G}{\partial\xi\,\partial\zeta}-\frac{\partial G}{\partial\xi}\frac{\partial^2 G}{\partial\zeta^2}\right)\right\}; \quad (3.5.4)$$

$$G,\ \partial G/\partial\zeta\to 0,\quad \partial^2 G/\partial\zeta^2\to\Gamma^2\quad \text{as}\quad \zeta\to 0,\quad G\to\infty\quad\text{as}\quad \zeta\to\infty. \quad (3.5.5)$$

From (3.5.4) and (3.5.5), it is determined that the first integral of the wall-layer equation is

$$\{1+\kappa\zeta N_0 M_0\}\frac{\partial^2 G}{\partial\zeta^2}-\left\{\Gamma^2+\frac{\Pi\zeta}{\xi S}\right\}+\frac{S^2}{\xi S}\frac{1-\Pi}{1+T}\left\{\xi\frac{\partial G}{\partial\xi}\frac{\partial G}{\partial\zeta}\right\}$$
$$=\frac{S^2}{\xi S}\frac{1}{1+T}\left\{[1-\Pi]\,\xi\frac{\partial}{\partial\xi}\left(\int_0^\zeta\left[\frac{\partial G}{\partial\zeta}\right]^2 d\zeta\right)-[\Pi+T]\left(\int_0^\zeta\left[\frac{\partial G}{\partial\zeta}\right]^2 d\zeta\right)\right\}. \quad (3.5.6)$$

3.6. $S(\xi)=\kappa/\log\xi$

Suppose that the defect-layer function and/or the wall-layer function $S(\xi)$ is given by

$$S(\xi)=\kappa/\log\xi,\quad \text{so that}\quad S(\xi)\to 0,\quad \xi\{S(\xi)\}^k\to\infty\quad\text{as}\quad \xi\to\infty. \quad (3.6.1)$$

Note that consideration of this case is suggested by the results obtained in the analysis of turbulent channel flow in §2 (cf., for example, (2.4.7), (2.5.6) and (2.6.1)).

3.6.1. *The defect layer*. For this 'distinguished limit', i.e.,

$$T(\xi)=-\xi S'(\xi)/S(\xi)=1/\log\xi\to 0\quad\text{as}\quad \xi\to\infty,$$

the defect-layer boundary-value problem for $f(\xi,\eta)=\kappa F(\xi,\eta)$ reduces to

$$\frac{\partial}{\partial\eta}\left(\left\{\kappa\eta M_0 N_0+\frac{1}{\xi}\right\}\frac{\partial^2 f}{\partial\eta^2}+[1-\Pi]\,\eta\frac{\partial f}{\partial\eta}-\frac{1}{1+T}\left\{[1-\Pi]\,\xi\frac{\partial f}{\partial\xi}+[(1-3\Pi)+T(-2\Pi)]f\right\}\right)$$
$$=\frac{T}{1+T}\left\{[1-\Pi]\frac{\partial}{\partial\eta}\left(\left[\xi\frac{\partial f}{\partial\xi}+f\right]\frac{\partial f}{\partial\eta}\right)\right.$$
$$\left.-[1-\Pi]\,\xi\frac{\partial}{\partial\xi}\left(\left[\frac{\partial f}{\partial\eta}\right]^2\right)-[(1-2\Pi)+T(-1)]\left[\frac{\partial f}{\partial\eta}\right]^2\right\}; \quad (3.6.2a)$$

$$f\to f_{\eta\to\infty}=\kappa\Lambda,\quad \frac{\partial f}{\partial\eta},\frac{\partial^2 f}{\partial\eta^2}\to 0\quad\text{as}\quad \eta\to\infty. \quad (3.6.2b)$$

With η fixed, $\xi\to\infty$, for this case, it is assumed, subject to verification, that the defect-layer stream function $f(\xi,\eta)$ and the pressure-gradient function $\Pi(\xi)$ have the following asymptotic expansions:

$$f(\xi,\eta)\cong\{f_0(\xi,\eta)+T(\xi)f_1(\xi,\eta)+T^2(\xi)f_2(\xi,\eta)+\dots\}+\dots; \quad (3.6.3a)$$

$$\Pi(\xi)\cong\{\Pi_0(\xi)+T(\xi)\,\Pi_1(\xi)+T^2(\xi)\,\Pi_2(\xi)+\dots\}+\dots. \quad (3.6.3b)$$

For such a formulation, it is determined that the equations and boundary conditions for $f_0(\xi,\eta)$ and $f_1(\xi,\eta)$ are

$$\kappa\eta M_0\frac{\partial^2 f_0}{\partial\eta^2}+[1-\Pi_0]\,\eta\frac{\partial f_0}{\partial\eta}-\left\{[1-\Pi_0]\,\xi\frac{\partial f_0}{\partial\xi}+[1-3\Pi_0]f_0\right\}$$
$$=-\left\{[1-\Pi_0]\,\xi\frac{df_{0,\infty}}{d\xi}+[1-3\Pi_0]f_{0,\infty}\right\}, \qquad (3.6.4a)$$

$$f_0\to f_{0,\infty},\quad \partial f_0/\partial\eta\to 0\quad\text{as}\quad \eta\to\infty; \qquad (3.6.4b)$$

$$\kappa\eta M_0\frac{\partial^2 f_1}{\partial\eta^2}+[1-\Pi_0]\,\eta\frac{\partial f_1}{\partial\eta}-\left\{[1-\Pi_0]\,\xi\frac{\partial f_1}{\partial\xi}+[1-3\Pi_0]f_1\right\}$$
$$=-\left\{[1-\Pi_0]\,\xi\frac{df_{1,\infty}}{d\xi}+[1-3\Pi_0]f_{1,\infty}\right\}$$
$$+\left\{[\Pi_1+(1-\Pi_0)]\,\xi\frac{df_{0,\infty}}{d\xi}+[3\Pi_1+(1-\Pi_0)]f_{0,\infty}\right\}$$
$$+\left\{[1-\Pi_0]\,\xi\frac{\partial}{\partial\xi}\left(\int_\eta^\infty\left[\frac{\partial f_0}{\partial\hat\eta}\right]^2 d\hat\eta\right)+[1-2\Pi_0]\left(\int_\eta^\infty\left[\frac{\partial f_0}{\partial\hat\eta}\right]^2 d\hat\eta\right)\right\}$$
$$+\Pi_1\left\{\eta\frac{\partial f_0}{\partial\eta}-\xi\frac{\partial f_0}{\partial\xi}-3f_0\right\}+[1-\Pi_0]\left\{\left[\xi\frac{\partial f_0}{\partial\xi}+f_0\right]\left[\frac{\partial f_0}{\partial\eta}-1\right]\right\}, \qquad (3.6.5a)$$

$$f_1\to f_{1,\infty},\quad \partial f_1/\partial\eta\to 0\quad\text{as}\quad \eta\to\infty. \qquad (3.6.5b)$$

Attention is now directed to the defect-layer tangential-velocity function $\bar u = 1-S\,\partial F/\partial\eta = 1-T\,\partial f/\partial\eta$. It follows that, within the framework of this asymptotic analysis, this velocity function is represented by the expansion

$$\bar u \cong 1-T\left\{\frac{\partial f_0}{\partial\eta}+T\frac{\partial f_1}{\partial\eta}+T^2\frac{\partial f_2}{\partial\eta}+\ldots\right\}+\ldots. \qquad (3.6.6)$$

Near the wall, as $\eta\to 0$, it is determined, from (3.6.4) and (3.6.5), that $\partial f_0/\partial\eta$ and $\partial f_1/\partial\eta$ have the following asymptotic expansions:

$$\frac{\partial f_0}{\partial\eta}\cong Q_0\left[\left\{\log\left(\frac{1}{\eta}\right)-I_0\right\}-\eta\left\{V_0\log\left(\frac{1}{\eta}\right)-W_0\right\}+\ldots\right], \qquad (3.6.7a)$$

where
$$\kappa Q_0(\xi) = [1-\Pi_0]\,\xi\frac{df_{0,\infty}}{d\xi}+[1-3\Pi_0]f_{0,\infty},$$
$$\kappa I_0(\xi) = \text{function of integration (to be determined)},$$
$$\kappa V_0(\xi) = 2\Pi_0-[1-\Pi_0]\left[\frac{\xi}{Q_0}\frac{dQ_0}{d\xi}\right],$$
$$\kappa W_0(\xi) = [1-3\Pi_0]-[1-\Pi_0]\left[\frac{\xi}{Q_0}\frac{dQ_0}{d\xi}\right](I_0-2)$$
$$+\left(2\Pi_0-[1-\Pi_0]\left[\frac{\xi}{I_0-1}\frac{dI_0}{d\xi}\right]\right)(I_0-1),\ldots; \qquad (3.6.7b)$$

and
$$\frac{\partial f_1}{\partial\eta}\cong Q_1\left[\left\{\log\left(\frac{1}{\eta}\right)-I_1\right\}-\eta\left\{V_1\log\left(\frac{1}{\eta}\right)-W_1\right\}+\ldots\right]$$
$$+Q_0^2\left\{\eta\log\left(\frac{1}{\eta}\right)\right\}\left[\left\{X_0\log\left(\frac{1}{\eta}\right)-Y_0\right\}+\ldots\right], \qquad (3.6.8a)$$

$$\text{where}\quad \kappa Q_1(\xi) = \left\{[1-\Pi_0]\,\xi\frac{df_{1,\infty}}{d\xi}+[1-3\Pi_0]f_{1,\infty}\right\}$$
$$-\left\{[\Pi_1+(1-\Pi_0)]\,\xi\frac{df_{0,\infty}}{d\xi}+[3\Pi_1+(1-\Pi_0)]f_{0,\infty}\right\}$$
$$-\left\{[1-\Pi_0]\,\xi\frac{d}{d\xi}\left(\int_0^\infty\left[\frac{\partial f_0}{\partial\eta}\right]^2 d\eta\right)+[1-2\Pi_0]\left(\int_0^\infty\left[\frac{\partial f_0}{\partial\eta}\right]^2 d\eta\right)\right\},$$
$$\kappa I_1(\xi) = \text{function of integration (to be determined)},\ldots. \tag{3.6.8b}$$

Thus, as $\eta\rightarrow 0$,

$$\bar{u} \cong 1-T[Q_0(\{\log(1/\eta)-I_0\}-\eta\{V_0\log(1/\eta)-W_0\}+\ldots)]$$
$$-T^2[Q_1(\{\log(1/\eta)-I_1\}+\ldots)+Q_0^2\{\eta\log(1/\eta)\}(\{X_0\log(1/\eta)-Y_0\}+\ldots)]-\ldots. \tag{3.6.9}$$

For the examination of the behaviour of $\bar{u}$ based on the defect-layer expansion, in the overlap domain, between the defect and wall layers, the intermediate normal variable λ, defined by

$$\lambda = \eta/\phi(\xi) = \zeta/\xi\phi(\xi);\quad \phi(\xi)\rightarrow 0,\quad \xi\phi(\xi)\rightarrow\infty \quad\text{as}\quad \xi\rightarrow\infty, \tag{3.6.10}$$

is introduced. Then, it follows that the defect-layer normal variable η obeys

$$\eta = \phi(\xi)\,\lambda\rightarrow 0\quad\text{for }\lambda\text{ fixed},\quad \xi\rightarrow\infty. \tag{3.6.11}$$

Application of the intermediate limit λ fixed, $\xi\rightarrow\infty$ to (3.6.9) yields

$$\bar{u}\cong 1-T[\{Q_0(\log\{1/\phi\lambda\}-I_0)\}+T\{Q_1(\log\{1/\phi\lambda\}-I_1)\}+\ldots]$$
$$+TQ_0(\phi\lambda)\,[(V_0\log\{1/\phi\lambda\}-W_0)-T\{Q_0(\log\{1/\phi\lambda\}-I_0)\,(X_0\log\{1/\phi\lambda\}-Y_0)\}+\ldots]+\ldots. \tag{3.6.12}$$

3.6.2. *The wall layer*. For $T = 1/\log\xi\rightarrow 0$ as $\xi\rightarrow\infty$, the wall-layer boundary-value problem for $g(\xi,\zeta) = \kappa G(\xi,\zeta)$ reduces to

$$\frac{\partial}{\partial\zeta}\left(\{1+\kappa\zeta N_0M_0\}\frac{\partial^2 g}{\partial\zeta^2}\right)-\frac{\Pi}{\xi T}$$
$$=\frac{T^2}{\xi T}\frac{1}{1+T}\left\{-[\Pi+T]\left[\frac{\partial g}{\partial\zeta}\right]^2+[1-\Pi]\,\xi\left(\frac{\partial g}{\partial\zeta}\frac{\partial^2 g}{\partial\xi\,\partial\zeta}-\frac{\partial g}{\partial\xi}\frac{\partial^2 g}{\partial\zeta^2}\right)\right\}; \tag{3.16.13}$$
$$g,\,\partial g/\partial\zeta\rightarrow 0\quad \partial^2 g/\partial\zeta^2\rightarrow\kappa\Gamma^2\quad\text{as}\quad\zeta\rightarrow 0.$$

With ζ fixed, $\xi\rightarrow\infty$, it is assumed, subject to verification, that the wall-layer stream function $g(\xi,\zeta)$ and the friction velocity and pressure-gradient functions $\Gamma(\xi)$ and $\Pi(\xi)$ have the following asymptotic expansions:

$$g(\xi,\zeta)\cong\{g_0(\xi,\zeta)+T(\xi)\,g_1(\xi,\zeta)+T^2(\xi)\,g_2(\xi,\zeta)+\ldots\}$$
$$+[1/\xi T(\xi)]\{g_{10}(\xi,\zeta)+T(\xi)\,g_{11}(\xi,\zeta)+T^2(\xi)\,g_{12}(\xi,\zeta)+\ldots\}+\ldots; \tag{3.6.14a}$$

$$\left.\begin{aligned}\Gamma^2(\xi)&\cong\{\Omega_0(\xi)+T(\xi)\,\Omega_1(\xi)+T^2(\xi)\,\Omega_2(\xi)+\ldots\}\\&\quad+[1/\xi T(\xi)]\{\Omega_{10}(\xi)+T(\xi)\,\Omega_{11}(\xi)+T^2(\xi)\,\Omega_{12}(\xi)+\ldots\}+\ldots,\\\Pi(\xi)&\cong\{\Pi_0(\xi)+T(\xi)\,\Pi_1(\xi)+T^2(\xi)\,\Pi_2(\xi)+\ldots\}+\ldots.\end{aligned}\right\} \tag{3.6.14b}$$

To leading orders of approximation, with $\tilde{\zeta} = \kappa\zeta$ (and $\tilde{N}_0(\tilde{\zeta}) = N_0(\zeta)$) and $\tilde{g}_i(\xi,\tilde{\zeta}) = \kappa g_i(\xi,\zeta)$, $i = 0,1,2,\ldots$, the flow in the wall layer is governed by

$$\{1+\tilde{\zeta}\tilde{N}_0\}\,\partial^2\tilde{g}_i/\partial\tilde{\zeta}^2 = \Omega_i;\quad \tilde{g}_i,\,\partial\tilde{g}_i/\partial\tilde{\zeta}\rightarrow 0\quad\text{as}\quad\tilde{\zeta}\rightarrow 0. \tag{3.6.15}$$

The first and second integrals of the above equations are given by

$$\partial\tilde{g}_i/\partial\tilde{\zeta} = \Omega_i\{\log(1+\tilde{\zeta})+\tilde{J}_1\} \equiv \Omega_i\, d\tilde{G}_0/d\tilde{\zeta}, \tag{3.6.16a}$$

$$\tilde{g}_i = \Omega_i\{(1+\tilde{\zeta})\log(1+\tilde{\zeta})+\tilde{\zeta}(\tilde{J}_1-1)-\tilde{J}_2\} \equiv \Omega_i\tilde{G}_0, \tag{3.6.16b}$$

where
$$\tilde{J}_k(\tilde{\zeta}) = \int_0^{\tilde{\zeta}} \frac{\tilde{\zeta}_1^k\{1-\tilde{N}_0(\tilde{\zeta}_1)\}}{\{1+\tilde{\zeta}_1\}\{1+\tilde{\zeta}_1\tilde{N}_0(\tilde{\zeta}_1)\}}\,d\tilde{\zeta}_1, \quad \text{with} \quad k = 1,2,\ldots. \tag{3.6.17}$$

From a consideration of the next higher orders of approximation, it is determined that the functions $\tilde{g}_{1i}(\xi,\tilde{\zeta}) = \kappa g_{1i}(\xi,\zeta)$, $i = 0, 1$, satisfy

$$\{1+\tilde{\zeta}\tilde{N}_0\}\frac{\partial^2\tilde{g}_{1i}}{\partial\tilde{\zeta}^2} = \Omega_{1i}+\left(\frac{1}{\kappa}\right)^2\Pi_1\tilde{\zeta}; \quad \tilde{g}_{1i}, \frac{\partial\tilde{g}_{1i}}{\partial\tilde{\zeta}} \to 0 \quad \text{as} \quad \tilde{\zeta}\to 0, \tag{3.6.18a}$$

so that

$$\frac{\partial\tilde{g}_{1i}}{\partial\tilde{\zeta}} = \Omega_{1i}\{\log(1+\tilde{\zeta})+\tilde{J}_1\}+\left(\frac{1}{\kappa}\right)^2\Pi_i\{\tilde{\zeta}-\log(1+\tilde{\zeta})+\tilde{J}_2\} \equiv \Omega_{1i}\frac{d\tilde{G}_0}{d\tilde{\zeta}}+\left(\frac{1}{\kappa}\right)^2\Pi_i\frac{d\tilde{G}_{10}}{d\tilde{\zeta}}. \tag{3.6.18b}$$

Further, it is determined that the $\tilde{g}_{1i}(\xi,\tilde{\zeta}) = \kappa g_{1i}(\xi,\zeta)$, $i = 2, 3, \ldots$, satisfy

$$\{1+\tilde{\zeta}\tilde{N}_0\}\frac{\partial^2\tilde{g}_{1i}}{\partial\tilde{\zeta}^2} = \Omega_{1i}+\left(\frac{1}{\kappa}\right)^2\Pi_i\tilde{\zeta}+\tilde{D}_i; \quad \tilde{D}_i = \tilde{D}_i(\xi,\tilde{\zeta}). \tag{3.6.19}$$

Note that the function $\tilde{D}_2$, for example, is expressible as

$$\tilde{D}_2 = \left(\frac{1}{\kappa}\right)^2\Omega_0^2\left\{-\Pi_0\left(\int_0^{\tilde{\zeta}}\left[\frac{d\tilde{G}_0}{d\tilde{\zeta}}\right]^2 d\tilde{\zeta}\right)\right.$$
$$\left.+[1-\Pi_0]\left[\frac{\xi}{\Omega_0}\frac{d\Omega_0}{d\xi}\right]\left[2\left(\int_0^{\tilde{\zeta}}\left[\frac{d\tilde{G}_0}{d\tilde{\zeta}}\right]^2 d\tilde{\zeta}\right)-\left(\tilde{G}_0\frac{d\tilde{G}_0}{d\tilde{\zeta}}\right)\right]\right\}. \tag{3.6.20}$$

Attention is now directed to the wall-layer tangential-velocity function $\bar{u} = S\,\partial G/\partial\zeta = T\,\partial g/\partial\zeta$. Within the framework of this asymptotic analysis, this velocity function is represented by the expansion

$$\bar{u} \cong T\left[\left\{\frac{\partial g_0}{\partial\zeta}+T\frac{\partial g_1}{\partial\zeta}+T^2\frac{\partial g_2}{\partial\zeta}+\ldots\right\}+\frac{1}{\xi T}\left\{\frac{\partial g_{10}}{\partial\zeta}+T\frac{\partial g_{11}}{\partial\zeta}+T^2\frac{\partial g_{12}}{\partial\zeta}+\ldots\right\}+\ldots\right]. \tag{3.6.21}$$

Near the wall, as $\zeta = (1/\kappa)\tilde{\zeta}\to 0$, the asymptotic behaviour of $\bar{u}$ is given by

$$\bar{u} \cong \kappa\left(\zeta\left[1-\frac{\kappa C}{m+2}\zeta^{m+1}+\ldots\right]\right)\left(T\left[\{\Omega_0+T\Omega_1+\ldots\}+\frac{1}{\xi T}\{\Omega_{10}+T\Omega_{11}+\ldots\}+\ldots\right]\right)$$
$$+\kappa\left(\tfrac{1}{2}\zeta^2\left[1-\frac{2\kappa C}{m+3}\zeta^{m+1}+\ldots\right]\right)\frac{1}{\kappa\xi}[\{\Pi_0+T\Pi_1+\ldots\}+\ldots]+\ldots. \tag{3.6.22}$$

Far from the wall, as $\zeta = (1/\kappa)\tilde{\zeta}\to\infty$, the asymptotic behaviour of $\bar{u}$ is given by

$$\bar{u} \cong T\left[\{(\log\zeta+J_1)+\ldots\}\left\{(\Omega_0+T\Omega_1+\ldots)+\frac{1}{\xi T}(\Omega_{10}+T\Omega_{11}+\ldots)+\ldots\right\}\right.$$
$$+\left\{\zeta-\frac{1}{\kappa}(\log\zeta-J_2)+\ldots\right\}\left\{\frac{1}{\kappa}\frac{1}{\xi T}(\Pi_0+T\Pi_1+T^2\Pi_2+\ldots)+\ldots\right\}$$
$$\left.+\{\zeta\log\zeta(\log\zeta+2(J_1-2))+\ldots\}\left\{\frac{1}{\kappa}\frac{1}{\xi T}(T^2D_2+\ldots)+\ldots\right\}+\ldots\right], \tag{3.6.23}$$

where $J_1 = \tilde{J}_1(\infty)-\log(1/\kappa)$, $J_2 = \tilde{J}_2(\infty)+\log(1/\kappa), \ldots$ are constants;

$$D_2 = \Omega_0^2\{-\Pi_0+[1-\Pi_0][(\xi/\Omega_0)\,d\Omega_0/d\xi]\}, \ldots.$$

For the examination of the behaviour of $\bar{u}$ based on the wall-layer expansion, in the overlap domain, between the defect and wall layers, the intermediate normal variable λ, defined by

$$\lambda = \eta/\phi(\xi) = \zeta/\xi\phi(\xi); \quad \phi(\xi) \to 0, \quad \xi\phi(\xi) \to \infty \quad \text{as} \quad \xi \to \infty, \tag{3.6.24}$$

is re-introduced. Then, it follows that wall-layer normal variable ζ obeys

$$\zeta = \{\xi\phi(\xi)\}\lambda \to \infty \quad \text{for } \lambda \text{ fixed}, \quad \xi \to \infty. \tag{3.6.25}$$

Application of the intermediate limit λ fixed, $\xi \to \infty$ to (3.6.23) yields

$$\bar{u} \cong \Omega_0 - T[\{\Omega_0(\log(1/\phi\lambda) - J_1) - \Omega_1\} + T\{\Omega_1(\log(1/\phi\lambda) - J_1) - \Omega_2\} + \dots] + \phi\lambda[\{(1/\kappa)(D_2 + \Pi_0)\} + \dots] + \dots. \tag{3.6.26}$$

3.6.3. *Results*. From a comparison of (3.6.12) and (3.6.26), it is seen that the postulated defect-layer and wall-layer solutions for the velocity function $\bar{u}$ match when

$$\Omega_0 = 1, \quad \Omega_1 = I_0 - J_1, \quad \Omega_2 = (I_0 - J_1)(I_1 - J_1), \dots; \tag{3.6.27a}$$

$$Q_0 = \Omega_0 = 1, \quad Q_1 = \Omega_1 = (I_0 - J_1), \dots. \tag{3.6.27b}$$

The results of (3.6.27*a*), in turn, produce the following leading terms in the expansions for the friction velocity function:

$$\Gamma^2 \cong \{1 + (1/\log\xi)(I_0 - J_1) + (1/\log\xi)^2(I_0 - J_1)(I_1 - J_1) + \dots\} + \dots. \tag{3.6.28}$$

Further, the results of (3.6.27*b*), with respect to the displacement-thickness function, yield

$$\Lambda \cong \{\Lambda_0 + (1/\log\xi)\Lambda_1 + \dots\} + \dots, \tag{3.6.29}$$

where Λ_0 and Λ_1 are given by

$$[1 - \Pi_0]\xi\frac{d\Lambda_0}{d\xi} + [1 - 3\Pi_0]\Lambda_0 = 1; \tag{3.6.30a}$$

$$\begin{aligned}[1 - \Pi_0]\xi\frac{d\Lambda_1}{d\xi} &+ [1 - 3\Pi_0]\Lambda_1 \\ &= (I_0 - J_1) + \left\{[\Pi_1 + (1 - \Pi_0)]\xi\frac{d\Lambda_0}{d\xi} + [3\Pi_1 + (1 - \Pi_0)]\Lambda_0\right\} \\ &\quad + \frac{1}{\kappa}\left\{[1 - \Pi_0]\xi\frac{d}{d\xi}\left(\int_0^\infty \left[\frac{\partial f_0}{\partial\eta}\right]^2 d\eta\right) + [1 - 2\Pi_0]\left(\int_0^\infty \left[\frac{\partial f_0}{\partial\eta}\right]^2 d\eta\right)\right\}.\end{aligned} \tag{3.6.30b}$$

As a direct consequence of the above, it is determined that in the present formulation the (preliminary) expression for the skin-friction coefficient c_f is

$$\begin{aligned}c_f = 2\frac{\nu^*(\partial u^*/\partial y^*)_{y=0}}{U^{*2}} = \frac{2}{Re}\frac{(\partial u/\partial y)_{y=0}}{U^2} &= 2(u_\tau/U)^2 = 2\gamma^2 = 2(S\Gamma)^2 = (2\kappa^2)(T\Gamma)^2 \\ &\cong (2\kappa^2)(1/\log\xi)^2[\{1 + (1/\log\xi)(I_0 - J_1) + (1/\log\xi)^2(I_0 - J_1)(I_1 - J_1) + \dots\} + \dots].\end{aligned} \tag{3.6.31}$$

Since $S(\xi)$ is given by

$$S(\xi) = \kappa T(\xi) = \kappa(1/\log\xi), \tag{3.6.32a}$$

it follows that $S(x)$ is given by

$$\begin{aligned}S(x) = \kappa T(x) &= \kappa(1/\log[Re\, U(x)\Delta(x)S(x)]) \\ &= \kappa(1/\log[\kappa\{Re\, U(x)\Delta(x)\}T(x)]).\end{aligned} \tag{3.6.32b}$$

With $Re^*_\Delta(x) \equiv \{Re\, U(x)\Delta(x)\}$,† then,

$$(1/T) + \log(1/T) = \log(\kappa Re^*_\Delta), \tag{3.6.33}$$

† Compare the role of $Re^*_\Delta = U^*(x^*)\Delta^*(x^*)/\nu^*$ in boundary-layer flow with that of $\overline{Re} = U^*h^*/\nu^*$ in channel flow.

so that
$$(1/T) \cong \log(\kappa Re^*_\Delta)\left[1-\frac{\log\{\log(\kappa Re^*_\Delta)\}}{\log(\kappa Re^*_\Delta)}+\ldots\right]; \qquad (3.6.34a)$$
$$S = \kappa T = d\Delta/dx \cong \frac{\kappa}{\log(\kappa Re^*_\Delta)}\left[1+\frac{\log\{\log(\kappa Re^*_\Delta)\}}{\log(\kappa Re^*_\Delta)}+\ldots\right]. \qquad (3.6.34b)$$

It is seen that (3.6.33) demonstrates that the two-layer asymptotic analysis of turbulent boundary-layer flow, as postulated in this subsection, is valid in the limit Re^*_Δ, $\log Re^*_\Delta \to \infty$. In turn, on the basis of the above, it is determined that the thickness gradient function $A(x) = \{(x/\Delta)\,d\Delta/dx\}^{-1}$ and the pressure-gradient function
$$\Pi(x) = -\{(x/\Delta)\,d\Delta/dx\}^{-1}(x/U)\,dU/dx \equiv -A(x)\,\Phi(x)$$
have the expansions
$$A \cong \{A_0+TA_1+\ldots\},\quad \Pi \cong \{\Pi_0+T\Pi_1+\ldots\} = -\Phi\{A_0+TA_1+\ldots\}, \qquad (3.6.35a)$$
where $T(x)$ is given by (3.6.32)–(3.6.34) and
$$A_0 = 1,\quad A_1 = (1/x)\left(\textstyle\int[1+\Phi]\,dx\right),\ldots. \qquad (3.6.35b)$$

Consider now the quantities $\gamma(x) = S(\xi)\,\Gamma(\xi)$ and $\delta(x)/\Delta(x) = S(\xi)\,\Lambda(\xi)$. The expansion for $\gamma(x)$ (cf. (3.6.27a) *et seq.*) is
$$\gamma \cong (\kappa T)\{\gamma_0+T\gamma_1+\ldots\};\quad \gamma_0 = \Omega_0 = 1,\quad \gamma_1 = \tfrac{1}{2}\Omega_1 = \tfrac{1}{2}(I_0-J_1),\ldots. \qquad (3.6.36)$$
Similarly, the expansion for $\delta(x)/\Delta(x)$ (cf. (3.6.27b) *et seq.*) is
$$(\delta/\Delta) \cong (\kappa T)\{(\delta/\Delta)_0+T(\delta/\Delta)_1+\ldots\};$$
$$x\,d(\delta/\Delta)_0/dx+[1+3\Phi]\,(\delta/\Delta)_0 = 1,$$
$$x\frac{d(\delta/\Delta)_1}{dx}+[1+3\Phi]\left(\frac{\delta}{\Delta}\right)_1 = \left\{(\Omega_1-A_1)-\left(x\frac{dA_1}{dx}+2A_1\right)\left(\frac{\delta}{\Delta}\right)_0\right\}$$
$$+\left(\frac{1}{\kappa}\right)\left\{x\frac{d}{dx}\left(\int_0^\infty\left[\frac{\partial f_0}{\partial\eta}\right]^2 d\eta\right)+[1+2\Phi]\left(\int_0^\infty\left[\frac{\partial f_0}{\partial\eta}\right]^2 d\eta\right)\right\},\ldots, \qquad (3.6.37a)$$
so that
$$\left(\frac{\delta}{\Delta}\right)_0 = \left\{\frac{1}{xU^3}\right\}\left\{\int U^3\,dx\right\} = \left\{\left(\frac{x}{Z}\right)\left(\frac{dZ}{dx}\right)\right\}^{-1};\quad Z = \int U^3\,dx;\ldots. \qquad (3.6.37b)$$

Note that from the above it follows that
$$\frac{\delta/\Delta}{\gamma} = \frac{\delta/\gamma}{\Delta} \cong \left(\frac{\delta}{\Delta}\right)_0\left\{1+T\left(\left[\left(\frac{\delta}{\Delta}\right)_1\Big/\left(\frac{\delta}{\Delta}\right)_0\right]-\gamma_1\right)+\ldots\right\}. \qquad (3.6.38)$$

With (3.6.31)–(3.6.34) as a background, it is now possible to present expressions for the skin-friction coefficient c_f in terms of the (more 'conventional') Reynolds-number functions $Re^*_\Delta = Re\,U\Delta = U^*\Delta^*/\nu^*$ and $Re^*_x = \{Re\,Ux\} = \{U^*x^*/\nu^*\}$.

(i) In terms of the former function, c_f is given by
$$c_f \cong \frac{2\kappa^2}{\{\log(\kappa Re^*_\Delta)\}^2}\left[1+\frac{2\log\{\log(\kappa Re^*_\Delta)\}+\Omega_1}{\log(\kappa Re^*_\Delta)}\right] \quad \text{as} \quad \log Re^*_\Delta \to \infty. \qquad (3.6.39a)$$

(ii) In terms of the latter function, c_f is given by
$$c_f \cong \frac{2\kappa^2}{\{\log(\kappa^2 Re^*_x)\}^2}\left[1+\frac{4\log\{\log(\kappa^2 Re^*_x)\}+\Omega_1}{\log(\kappa^2 Re^*_x)}+\ldots\right] \quad \text{as} \quad \log Re^*_x \to \infty. \qquad (3.6.39b)$$

In order to compare the results of this analysis with existing analyses (e.g., Mellor & Gibson 1966), it is appropriate to introduce (i) the displacement-thickness Reynolds number (function) $\tilde{R}^* \equiv Re_\delta^* = U^*\delta^*/\nu^*$ and (ii) the pressure-gradient function $\tilde{\beta} = \delta^*(dP^*/dx^*)/\rho^* u_\tau^{*2}$. In terms of quantities introduced in this analysis, including $Z \equiv \int U^3\,dx$, these functions are given by

$$\tilde{R}^* = \xi\Delta = (\delta/\Lambda)\,Re_\Delta^* = \{(\delta/\Delta)\,(\Delta/x)\}\,Re_x^*$$

$$\cong \left\{\frac{x}{Z}\frac{dZ}{dx}\right\}^{-1}\frac{\kappa\,Re_\Delta^*}{\{\log(\kappa\,Re_\Delta^*)\}^2}+\ldots \cong \left\{\frac{x}{Z}\frac{dZ}{dx}\right\}^{-1}\frac{\kappa^2\,Re_x^*}{\{\log(\kappa^2 Re_x^*)\}^2}+\ldots, \qquad (3.6.40a)$$

$$\tilde{\beta} = \Pi\Lambda/\Gamma^2 = \left\{\frac{\Pi}{\gamma/S}\right\}\left\{\frac{\delta/\gamma}{\Delta}\right\} \cong -\frac{1}{3}\left\{Z\left(\frac{dZ}{dx}\right)^{-2}\frac{d^2Z}{dx^2}\right\}+\ldots. \qquad (3.6.40b)$$

Further, the normal spatial variables of the previous analyses are related to those of this analysis by

$$\tilde{\eta} \equiv \frac{u_\tau^*}{U^*}\frac{y^*}{\delta^*} = \left\{\frac{\delta/\gamma}{\Delta}\right\}^{-1}\eta \cong \left\{\frac{x}{Z}\frac{dZ}{dx}\right\}\eta+\ldots; \qquad (3.6.41a)$$

$$\tilde{\zeta} \equiv \tilde{R}^*\tilde{\eta} \equiv u_\tau^* y^*/\nu^* = (\gamma/S)\,\zeta \cong \zeta+\ldots. \qquad (3.6.41b)\dagger$$

In turn, with $\xi = \tilde{R}^*/\Upsilon\Gamma$, $\eta = \Upsilon\tilde{\eta}$, $\zeta = \tilde{\zeta}/\Gamma$, where

$$\Upsilon \equiv \Lambda/\Gamma = (\delta/\Delta)/\gamma = (\delta/\gamma)/\Delta \to \{(x/Z)\,dZ/dx\}^{-1}$$

(cf. equation (3.6.38)) and $\Gamma = \gamma/S \to 1$ (cf. equation (3.6.36)), the eddy viscosity of (3.2.1) may be re-expressed as

$$\epsilon = \{\tilde{R}^*/\Upsilon\Gamma\}\,M(\Upsilon\tilde{\eta})\,N(\tilde{\zeta}/\Gamma)$$

$$= \{\tilde{R}^*/\Gamma\}\,[\kappa\tilde{\eta}M_0(\Upsilon\tilde{\eta})]\,N_0(\{\tilde{R}^*/\Gamma\}\,\tilde{\eta}) = [\kappa(\tilde{\zeta}/\Gamma)\,N_0(\tilde{\zeta}/\Gamma)]\,M_0(\{\tilde{R}^*/\Upsilon\}^{-1}\tilde{\zeta}), \qquad (3.6.42a)$$

so that, in the defect and wall layers, respectively,

$$\epsilon \to \tilde{R}^*[\kappa\tilde{\eta}M_0(\Upsilon\tilde{\eta})]; \quad \epsilon \to [\kappa\tilde{\zeta}N_0(\tilde{\zeta})]. \qquad (3.6.42b)$$

Note that since $\Pi = (\Gamma/\Upsilon)\,\tilde{\beta} \to (1/\Upsilon)\,\tilde{\beta}$ (cf. equation (3.6.40*b*)), with some mathematical manipulation, it may be determined that Υ is related to $\tilde{\beta}$ by

$$x\frac{d\Upsilon}{dx}+\Upsilon \to [1+3\tilde{\beta}], \quad \text{so that} \quad \Upsilon \to 1+\frac{3}{x}\int\tilde{\beta}\,dx. \qquad (3.6.43)$$

To complete the picture of the two-layer turbulent boundary layer, consider the uniformly valid (or composite) expansion for the tangential-velocity function $\bar{u} = u/U$. On the basis of the defect-layer and the wall-layer expansions (3.6.6) and (3.6.21) for this function, and the matching conditions (3.6.27) on these expansions, it is determined that the composite expansion (cf. for example, Van Dyke 1964; Cole 1968) for $\bar{u}$, to leading orders of approximation, is given by

$$\bar{u} \cong T\left\{\left(\frac{\partial g_0}{\partial\zeta}-\left[\frac{\partial f_0}{\partial\eta}-\left(\log\frac{1}{\eta}-I_0\right)\right]\right)\right.$$

$$\left.+T\left(\frac{\partial g_1}{\partial\zeta}-\left[\frac{\partial f_1}{\partial\eta}-(I_0-J_1)\left(\log\frac{1}{\eta}-I_1\right)\right]\right)+\ldots\right\}+\ldots. \qquad (3.6.44)$$

† The variable $\tilde{\zeta}$ defined in (3.6.41*b*) is, of course, not to be confused with the $\tilde{\zeta}$ introduced below (3.6.14).

For the purposes of comparison, it should be noted that Coles (1969) has proposed a 'composite' formulation for $\bar{u}$, based on a survey of existing experimental data. This formulation, written in terms of the variables of the present analysis, is given by

$$\bar{u} \doteqdot T\{\Gamma(\tilde{f}+\tilde{\pi}\tilde{\omega})\}, \tag{3.6.45}$$

where $\tilde{\pi}$ is a function of ξ, $\tilde{\omega}$ is a function of $(\Delta/\tilde{\delta})\,\eta$, $\tilde{\delta}$ being the 'boundary-layer thickness function', and $\tilde{f}$ is a function of $\Gamma\zeta$ (see Coles (1969) for the properties of these functions). If attention is limited to the lowest order results in (3.6.44) and (3.6.45) (such that $\Gamma \to 1$), then (i) $\tilde{f}$ is identifiable with $\partial g_0/\partial\zeta$, and, consequently, (ii) $\tilde{\pi}\tilde{\omega}$ must be identified with (though it is not as general as)

$$-[\partial f_0/\partial\eta - (\log(1/\eta) - I_0)].$$

3.6.4. *Concluding remarks.* In this section, then, for $S(\xi) = \kappa(1/\log\xi)$, an asymptotic analysis of the turbulent boundary-layer boundary-value problem (3.3.3)–(3.3.6) for a 'sufficiently general' eddy-viscosity model, (3.2.1) with (3.2.2), has developed the 'wall velocity law' and higher order corrections (cf. equation (3.6.21)), and the 'defect velocity law' and higher order corrections (cf. equation (3.6.6)).

It is noted that the current analysis is a quite general one, in that, for the determination of the solutions for the outer (or defect-layer) expansions and the inner (or wall-layer) expansions of flow quantities, it is not required that these solutions be self-similar. Previous analyses (cf. Mellor & Gibson 1966) have treated the less general case of the (so-called) equilibrium turbulent boundary layer, in which, to at least the leading orders of approximation, the solutions for the flow quantities in the outer and inner layers are required to be self-similar.

(i) With $\Omega_0 = 1$ (cf. equation (3.6.27*a*)), it is seen that, to lowest order, the wall layer is self-similar, since $g_0(\xi,\zeta) = g_E(\zeta) = (1/\kappa)\,\tilde{G}_0(\kappa\zeta)$ (cf. equation (3.6.16)).

(ii) From (3.6.4) and (3.6.7), it is seen that, to lowest order, the defect layer is also self-similar if $f_0(\xi,\eta) = f_E(\eta)$, which, in turn, requires that $\Pi_0(\xi) = \Pi_E =$ constant. This requirement on the pressure-gradient function for self-similar behaviour in the defect layer corresponds to that proposed by Clauser (1956) from experimental observation. The constancy of Π_0 implies a power-law dependence of the outer edge speed U on the tangential co-ordinate x. The special properties of equilibrium (or self-similar) flow have been fully developed so often in the past that repetition here is unnecessary (see Mellor & Gibson (1966) for a discussion of the literature). It should be noted, however, that, with

$$\Upsilon_E = \{(\delta/\gamma)/\Delta\}_E \to 1/[1-3\Pi_E] \to [1+3\tilde{\beta}_E]$$

(cf. equation (3.6.43)), the frequently adopted identification of the boundary-layer thickness Δ_E with $(\delta/\gamma)_E$ is not exact, even to leading order of approximation, except for the special case of the zero-pressure-gradient boundary layer, where $\tilde{\beta}_E \equiv 0$.

It should be noted that once the conditions for matching are given (cf. equation (3.6.27)), for the wall layer, to the orders of approximation considered, the (detailed) velocity profiles can be obtained, subject to the introduction of a specific function $N_0(\zeta)$ to perform the indicated quadratures (cf. equations

(3.6.16)–(3.6.20)). The analogous statement for the defect layer, however, is that, with these matching conditions given for the defect layer, the (detailed) velocity profiles (to successive orders of approximation) can be obtained, subject (now) to the introduction of a specific function $M_0(\eta)$, from a sequence of solutions to (properly posed) two-point boundary-value problems (cf. equations (3.6.4)–(3.6.8)). These defect-layer boundary-value problems may be treated by numerical or weighted residual techniques, with the computations cutting off at the inner edge of this layer (rather than continuing through the wall layer to the wall itself).

The authors wish to acknowledge, with gratitude, helpful discussions with Dr R. Chapkis and Dr T. Kubota. This work was initiated under contract DAHC04-67-C0015 with the U.S. Army Research Office (Durham, North Carolina). The completion of the work was sponsored by Project SQUID, which is supported by the Office of Naval Research, Department of the Navy, under contract N00014-67-A-0226-0005, NR-098-038. Reproduction in part or in full is permitted for any use of the United States Government.

Appendix. Turbulent channel flow: alternative eddy-viscosity model

The non-dimensional boundary-value problem for turbulent channel flow† is taken to be (cf. (2.1.5) and (2.1.7))

$$\frac{1}{Re}\{1+\epsilon\}\frac{du}{dy}+u_\tau^2 y = 0; \tag{A 1a}$$

$$u \to 0, \quad \epsilon \to 0 \quad \text{as} \quad y \to 1, \quad u \to 1, \quad \frac{1}{Re}\{1+\epsilon\}\frac{du}{dy} \to 0 \quad \text{as} \quad y \to 0. \tag{A 1b}$$

The eddy-viscosity model adopted here (for the closure of (A 1)) is one based on mixing-length theory, namely,

$$\epsilon(\eta,\zeta;\xi) = \xi\hat{M}(\eta)\,\hat{N}(\xi\eta)\left\{\frac{1}{u_\tau(\xi)}\frac{du(\eta;\xi)}{d\eta}\right\} = \xi\hat{N}(\zeta)\,\hat{M}\left(\frac{\zeta}{\xi}\right)\left\{\frac{\xi}{u_\tau(\xi)}\frac{du(\zeta;\xi)}{d\zeta}\right\}, \tag{A 2}$$

with $\xi = Re\,u_\tau(Re)$; $\eta = 1-y$, $\zeta = \xi\eta = Re\,u_\tau(Re)\{1-y\}$. It is assumed (again) that $u_\tau(Re) \to 0$ and $\xi(Re) = Re\,u_\tau(Re) \to \infty$ as $Re \to \infty$ (such that $u_\tau(\xi) \to 0$ as $\xi \to \infty$). The functions $\hat{M}(\eta)$ and $\hat{N}(\zeta)$ have the following asymptotic forms:‡

$$\left.\begin{aligned} &\hat{M}(\eta) = \{\hat{\kappa}\eta\hat{M}_0(\eta)\}^2 \to (\hat{\kappa}\eta)^2(1+\hat{\kappa}_2\eta^2+\ldots)\to 0 \quad \text{as} \quad \eta \to 0 \\ &\qquad\qquad (\hat{\kappa}_1, \hat{\kappa}_2, \ldots = \text{constants of } O(1)), \\ &\hat{M}(\eta) \to \hat{K}_1\{1-\eta\}^{-1}[1-O(\{1-\eta\}^n)] \to \infty_1 \quad \text{as} \quad \eta \to 1 \\ &\qquad\qquad (\hat{K}_1 = \text{constant of } O(1)); \end{aligned}\right\} \tag{A 3a}$$

$$\left.\begin{aligned} &\hat{N}(\zeta) = \hat{N}_0(\zeta) \to \hat{C}\zeta+\ldots \to 0 \quad \text{as} \quad \zeta \to 0 \quad (\hat{C}, \ldots = \text{constants of } O(1)). \\ &\hat{N}(\zeta) = \hat{N}_0(\zeta) \to [1-O(\exp\{-\zeta\})] \to 1 \quad \text{as} \quad \zeta \to \infty. \end{aligned}\right\} \tag{A 3b}$$

† The notation of the appendix is that of § 2, except where noted.

‡ The authors plan to pursue certain aspects of the model further. The analysis presented here for this model is limited to the leading order matching of the defect-layer and wall-layer solutions.

For the defect layer, the spatial and velocity variables are

$$\eta = 1-y, \quad f(\eta;\xi) = [1-u(y;Re)]/u_\tau(Re); \tag{A 4}$$

the boundary-value problem, in the domain $0 < \eta \leqslant 1$, is given by

$$\left.\begin{array}{c}\left\{(\hat{\kappa}\eta)^2\hat{M}_0^2(\eta)\hat{N}_0(\xi\eta)\left[-\dfrac{df(\eta;\xi)}{d\eta}\right]+\dfrac{1}{\xi}\right\}\left[-\dfrac{df(\eta;\xi)}{d\eta}\right]-\{1-\eta\} = 0;\\ f(\eta;\xi)\to 0 \quad \text{as} \quad \eta\to 1.\end{array}\right\} \tag{A 5}$$

With $f(\eta;\xi)$ taken to have an asymptotic expansion of the form

$$f(\eta;\xi) \cong f_0(\eta)+(1/\xi)f_1(\eta)+\ldots, \tag{A 6}$$

it is determined that

$$\left.\begin{array}{l}\{f_0'(\eta)\}^2 = (1-\eta)/\{\hat{\kappa}\eta\hat{M}_0(\eta)\}^2,\\ \text{so that} \quad f_0'(\eta) = -(1-\eta)^{\frac{1}{2}}/\hat{\kappa}\eta\hat{M}_0(\eta), \quad f_0(1) = 0, \ldots.\end{array}\right\} \tag{A 7}$$

The solution of the above equation for $f_0(\eta)$ is

$$f_0(\eta) = (1/\hat{\kappa})\,[f_{00}(\eta)-f_{01}(\eta)], \tag{A 8a}$$

where

$$f_{00}(\eta) = \frac{\log(1/\eta)-2\{(1-\eta)^{\frac{1}{2}}-\log[1+(1-\eta)^{\frac{1}{2}}]\}}{\hat{M}_0(\eta)}, \quad f_{01}(\eta) = \int_\eta^1 \frac{\hat{M}_0'(t)}{\hat{M}_0(t)}f_{00}(t)\,dt. \tag{A 8b}$$

Application of the intermediate limit (λ fixed, $\xi\to\infty$), as defined by (2.4.9) and (2.4.10), to the defect-layer velocity $u(\eta;\xi) \cong 1-u_\tau(\xi)[f_0(\eta)+(1/\xi)f_1(\eta)+\ldots]$ yields

$$u \cong 1-(1/\hat{\kappa})\,u_\tau\{[\log\{1/\phi\lambda\}-\hat{w}_0+\tfrac{1}{2}\phi\lambda+\ldots]+\ldots\}, \tag{A 9}$$

where $\hat{w}_0 = [2(1-\log 2)+f_{01}(0)], \ldots$ are constants.

For the wall layer, the spatial and velocity variables are

$$\zeta = \xi\eta = Re\,u_\tau(Re)\{1-y\}, \quad g(\zeta;\xi) = u(y;Re)/u_\tau(Re); \tag{A 10}$$

and the boundary-value problem, in the domain $0 \leqslant \zeta < \infty$, is given by

$$\left.\begin{array}{c}\left\{1+(\hat{\kappa}\zeta)^2\hat{N}_0(\zeta)\hat{M}_0^2(\zeta/\xi)\dfrac{dg(\zeta;\xi)}{d\zeta}\right\}\dfrac{dg(\zeta;\xi)}{d\zeta}-\{1-(\zeta/\xi)\} = 0;\\ g(\zeta;\xi)\to 0 \quad \text{as} \quad \zeta\to 0.\end{array}\right\} \tag{A 11}$$

With $g(\zeta;\xi)$ taken to have an asymptotic expansion of the form

$$g(\zeta;\xi) \cong g_0(\zeta)+(1/\xi)\,g_1(\zeta)+\ldots, \tag{A 12}$$

it is found that

$$\left.\begin{array}{l}\{1+(\hat{\kappa}\zeta)^2\hat{N}_0(\zeta)g_0'(\zeta)\}g_0'(\zeta) = 1,\\ \text{so that} \quad g_0'(\zeta) = \dfrac{[1+4(\hat{\kappa}\zeta)^2\hat{N}_0(\zeta)]^{\frac{1}{2}}-1}{2(\hat{\kappa}\zeta)^2\hat{N}_0(\zeta)}, \quad g_0(0) = 0, \ldots.\end{array}\right\} \tag{A 13}$$

The solution for $g_0(\zeta)$ is

$$g_0(\zeta) = (1/\hat{\kappa})\,[g_{00}(\zeta)+g_{01}(\zeta)], \tag{A 14a}$$

where

$$\left.\begin{array}{l}g_{00}(\zeta) = \log(1+\hat{\kappa}\zeta),\\ g_{01}(\zeta) = \hat{\kappa}\displaystyle\int_0^\zeta\left[\frac{\{1+4(\hat{\kappa}s)^2\hat{N}_0(s)\}^{\frac{1}{2}}-1}{2(\hat{\kappa}s)^2\hat{N}_0(s)}-\frac{1}{1+\hat{\kappa}s}\right]ds.\end{array}\right\} \tag{A 14b}$$

Application of the intermediate limit (cf. (2.5.8) and (2.5.9)) to the wall-layer velocity $u(\zeta;\xi) \cong u_\tau(\xi)\{g_0(\zeta)+(1/\xi)\,g_1(\zeta)+\dots\}$ yields

$$u \cong (1/\hat{\kappa})\,u_\tau\{[\log\xi-\log\{1/\phi\lambda\}+\hat{\jmath}_0+\dots]+\dots\}, \tag{A 15}$$

where $\hat{\jmath}_0 = -[\log(1/\kappa)-g_{01}(\infty)],\dots$ are constants.

From a comparison of (A 9) and (A 15), it is seen that the defect- and wall-layer solutions for the velocity match when

$$1 \cong (1/\hat{\kappa})\,u_\tau(\xi)\log\xi\{[1+\hat{c}_0/\log\xi]+\dots\}, \tag{A 16}$$

with $\hat{c}_0 = (\hat{\jmath}_0-\hat{w}_0),\dots$ constants (cf. results of (2.6.1)).

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